Numerical Simulation of Lasers

Christoph Pflaum

Contents

1 Basic Properties of a Laser

1.1 Elements of a Laser

A laser consists mainly of the following three elements (see Figure 1) :

1. Laser medium: collection of atoms, molecules, ions or a semiconductor crystal:
   - gas laser
   - solid state lasers
   - semiconductor lasers
   - fiber laser
2. Pumping process to excite the atoms (molecules) into higher quantum mechanical energy levels.
3. Suitable optical feedback elements
   - as a laser amplifier (one pass of the beam)
   - as a laser oscillator (bounce back and forth of the laser beam)

1. Population inversion (see Figure 19)
2. Amplification of light (electromagnetic radiation) within a certain narrow band of frequencies. The amplification depends on the population inversion.
3. Oscillation: There must be more gain than loss of the beam. Reasons of loss are:
   - loss by medium
   - not accurate construction of the mirrors
   - output
4. Eigenmodes of a laser (e.g. Gauss modes , see Figure 3).
   - deformation of the crystal
   - gain, lenses
   - different refraction index

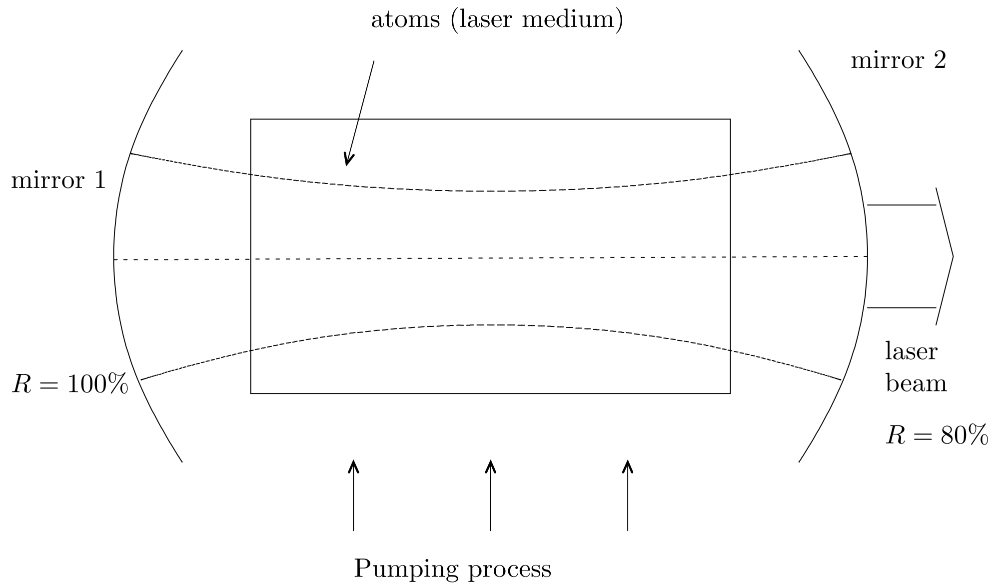


Figure 1: Basic Properties of a Laser

1.2 Atomic Energy Levels, Spontaneous Emission and Stimulated Transition

Light of a certain wavelength is emitted if a transition between two energy levels $E_2 \rightarrow E_1$ takes place
" jump of electrons " .

Formula 1. *The frequency of the emitted light is*

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}, \tag{1}$$

where

$$\hbar = \frac{h}{2\pi}, \quad h = 6.626 \cdot 10^{-34} Js \quad \textit{Planck's constant.}$$

Notation for wavelength: $1\mu m = 10000\,\mathring{A}$
Due to this formula, the energy levels can be described by

- $\frac{1}{\lambda}$ in cm^{-1} where λ is the wavelength of the corresponding wave and
- by a value with unit eV.

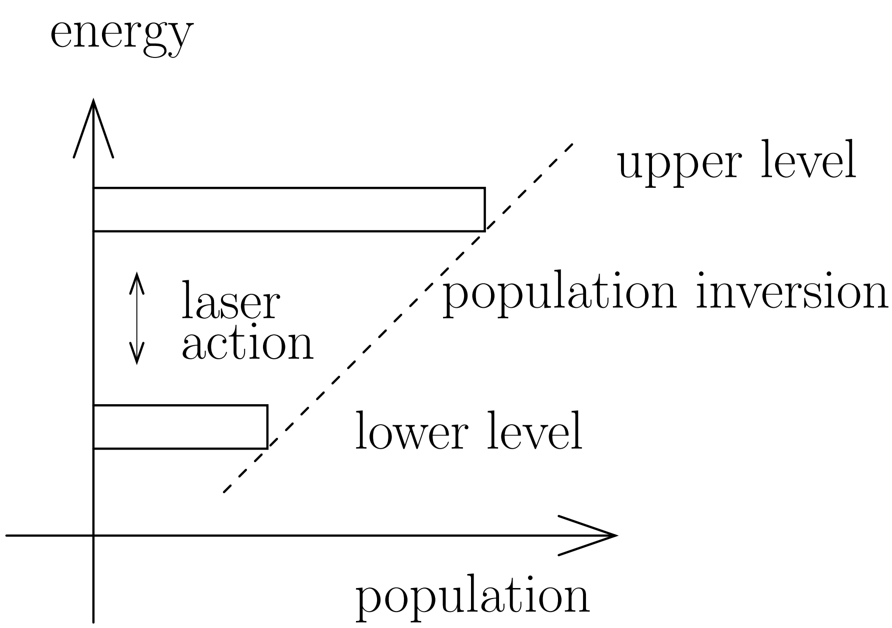


Figure 2: Population inversion

Transition from $E_2 \to E_1$ takes place
only with a little additional energy:

- spontaneous emission: energy from small movements of the atoms
- stimulated emission: energy from absorption

Let N_i be the number of atoms with energy level E_i.

Within a short period of time a certain percentage of atoms make a transition to a lower level.

This can be described by the following ODE:

$$\left.\frac{dN_2}{dt}\right|_{spon} = -\gamma N_2 = -\frac{N_2}{\tau},$$

where

- γ is called energy-decay rate and
- $\tau = \frac{1}{\gamma}$ is called lifetime.

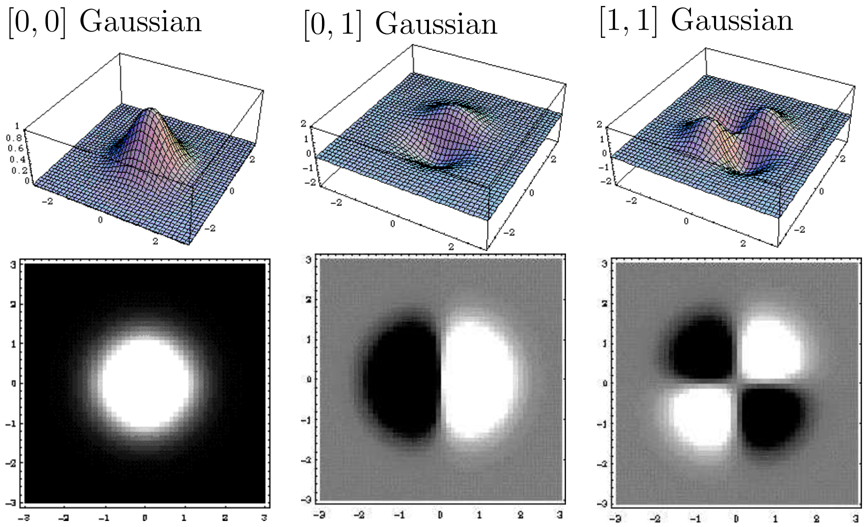


Figure 3: Hermite-Gaussian Modes

The solution of this ODE is:

$$N_2(t) = N_2(0)e^{-\frac{t}{\tau}}$$

If an external radiation signal is applied to the atom, then stimulated transitions occur: " atom reacts like an antenna ".
Let $n(t)$ be the photon density of the radiation.
Then, there is a constant K such that (see Figure) 4

$$\begin{aligned} \frac{dN_2}{dt}\Big|_{stim.upward} &= Kn(t)N_1(t), \quad \text{(absorption)} \\ \frac{dN_2}{dt}\Big|_{stim.downward} &= -Kn(t)N_2(t) \quad \text{(stimulated emission).} \end{aligned}$$

This implies:

$$\frac{dN_2}{dt}\Big|_{total} = Kn(t)(N_1(t) - N_2(t)) - \gamma_{21}N_2(t) = -\frac{dN_1}{dt}\Big|_{total}.$$

The total rate of signal stimulated transition between two energy levels is:

$$Kn(t) \cdot (N_1(t) - N_2(t)).$$

The energy transfer of stimulated transition by a signal is

$$\frac{dU_a}{dt} = Kn(t)(N_1(t) - N_2(t)) \cdot \hbar\omega,$$

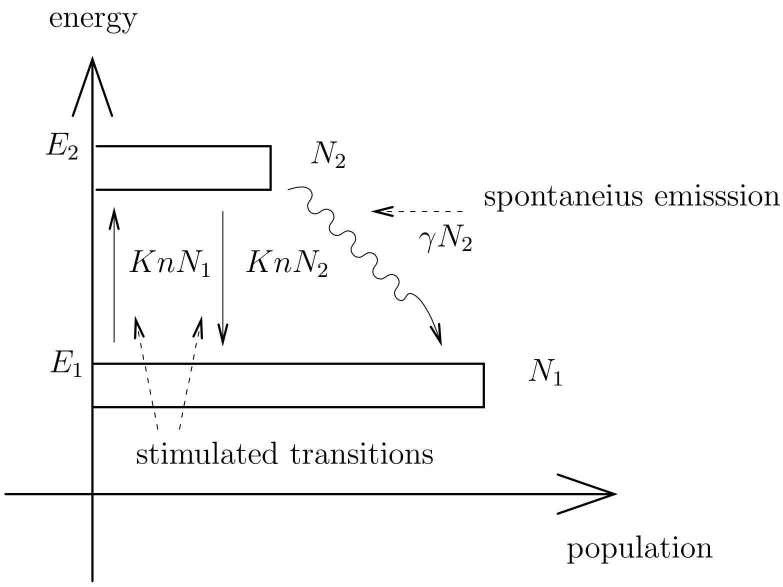


Figure 4: Stimulated transition

where U_a is the energy density.
The energy transfer changes the photon density of the signal by:

$$\frac{dn(t)}{dt} = -K(N_1(t) - N_2(t)) \cdot n(t). \tag{2}$$

- Absorption (attenuation): $N_1(t) > N_2(t)$
- Population inversion: $N_1(t) < N_2(t)$
  $\rightarrow$ net amplification of a signal

1.3 Pumping Process and Population Inversion

Population inversion means that

$$N_1 < N_2$$

where N_i is the number of atoms with energy level E_i, such that $E_2 > E_1$. In equilibrium there is no population inversion. The reason for this is Boltzmann's Principle of equilibrium:

Theorem 1 (Boltzmann's Principle). *In case of equilibrium the populations N_1 and N_2 depend on the temperature:*

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{kT}\right).$$

This implies

$$N_1 - N_2 = N_1\left(1 - e^{-\hbar\frac{\omega}{kT}}\right).$$

To obtain population inversion, a pumping process is needed, which destroys the state of equilibrium. Figure 5 shows a model of three level pumping process.

Let

- R_{p0} be the pumping rate (atoms/sec),
- η_p the pumping efficiency such that $R_p = \eta_p R_{p0}$ and
- γ_{ij} the decay rate from level E_i to E_j.

The following formulas describe the pumping process (without stimulated transitions):

$$\begin{aligned}
\frac{dN_2}{dt} &= R_p - \gamma_{21}N_2 \\
\frac{dN_1}{dt} &= \gamma_{21}N_2 - \gamma_{10}N_1
\end{aligned}$$

If $\frac{dN_i}{dt} = 0$, then we get

$$N_2 > N_1 \quad \text{(population inversion)} \quad \Leftrightarrow \quad \tau_{10} < \tau_{21}$$

1.4 Example of Scalar Rate Equations

Let us consider a four level pumping process according [7].

Let us abbreviate

$$N = N_2 - \frac{g_2 N_1}{g_1}$$

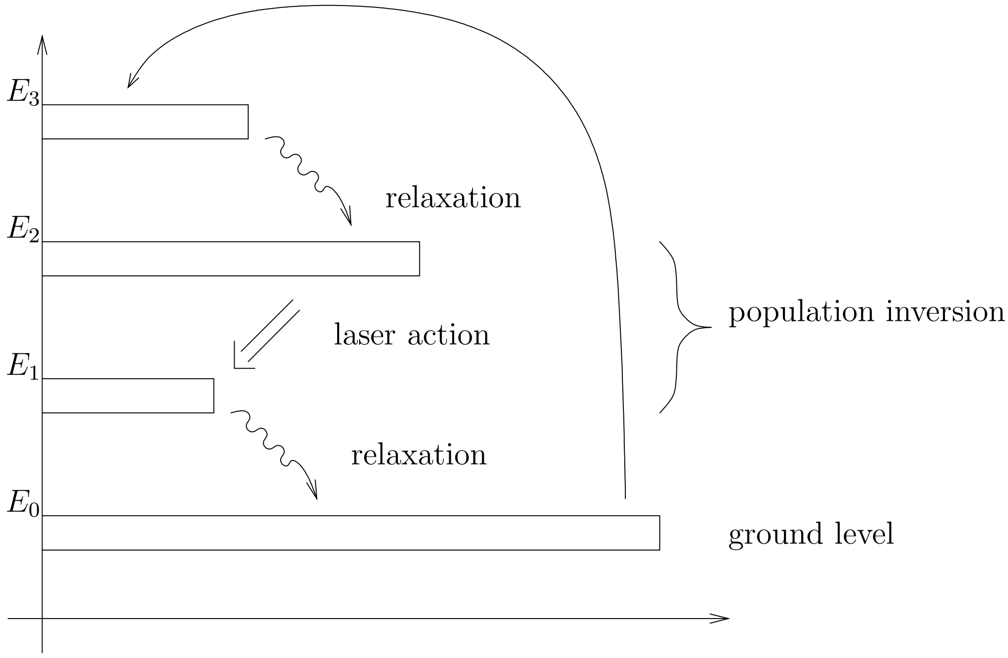


Figure 5: Three-level laser pumping process

then, the scalar rate equations are

$$\frac{\partial N}{\partial t} = -\gamma N n \sigma c - \frac{N + N_{tot}(\gamma - 1)}{\tau_f} + R_p(N_{tot} - N) \tag{3}$$

$$\frac{\partial n}{\partial t} = N n \sigma c - \frac{n}{\tau_c} + S. \tag{4}$$

The unknowns of these equations are

- N: population inversion $N = N_2 - \frac{g_2 N_1}{g_1}$.
- n: photon density

Parameters for Ruby are (see [7] section 2.2):

- g_1, g_2: degeneracy factors for quantum energy levels of Ruby:
  - $g(N_1) = 4$
  - $g(N_2(R_1)) = 2$, where R_1 is the green band with wavelength $6943\,\mathring{A}$

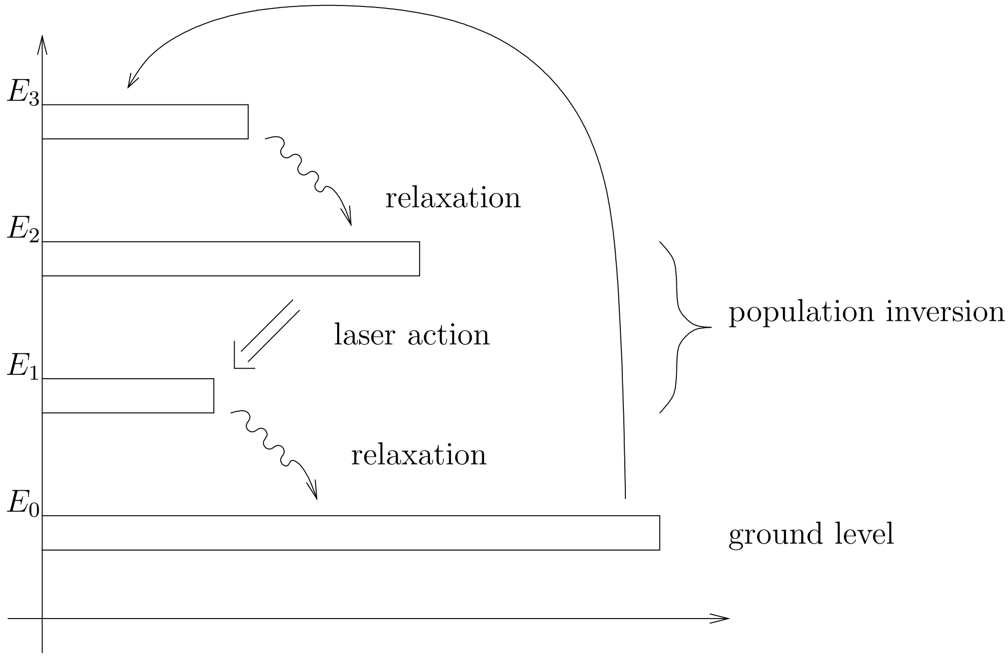


Figure 6: Four-level laser pumping process

  - $g(N_2(R_2)) = 2$, where R_2 is the blue band with wavelength 6929 $\mathring{A}$

- σ: stimulated emission cross section $\sigma_{21} = 2.5 \cdot 10^{-20} cm^2$
- N_{tot}: $1.58 \cdot 10^{19} ions/cm^3$ is the maximal population inversion.
- R_{p0}: pumping rate $atoms/sec$
- η_p: Quantum efficiency 0.7
- τ_f: $3ms$ (see page 15 in [7]) for R_1 line.
- S is a small value needed for the start up of a laser.
- τ_c: decay rate of photons.

1.5 Laser Amplification and Oscillation Condition

Let us assume that the optical wave can be modeled by

$$\tilde{E}(z,t) = \exp(j\omega t)E(z)$$

$$
\begin{aligned}
E(z) &= \exp(-jkz + \alpha_m z) = \exp(-jkz)u(z) \\
u(z) &= \exp(\alpha_m z).
\end{aligned}
$$

This implies that

$$\tilde{E}(z,t) = \exp(j\omega t) \;\cdot\; \exp(-jkz + \alpha_m z)$$

Thus, a constant phase shift is obtained at $\omega t = kz$.
Since $t = z/c$ in vacuum, we get

$$k = \frac{\omega}{c}.$$

(By $k^2 = \mu\epsilon\omega^2$ in Section 3, we obtain $c = \frac{1}{\sqrt{\mu\epsilon}}$ in vacuum.)

Now, let us model the optical wave by

$$
\begin{aligned}
\tilde{E}(z,t) &= \exp(j\omega t)E(z) \\
E(z) &= \exp(-j\omega z/c + \alpha_m z) = \exp(-j\omega z/c)u(z) \\
u(z) &= \exp(\alpha_m z).
\end{aligned}
$$

An increase of the photons leads to a gain of the optical wave:

$$|E(z)|^2 = |E_0|^2 \exp(+2\alpha_m z)$$

for the intensity of the optical wave, which is proportional to the photon density. Let r_i be the reflection coefficient at the mirrors $M_i, i = 1, 2$.
Let L_m be the length of the amplification medium.
Let L be the length of the laser medium.
Figure 7 shows one round trip of the optical wave.

Then, the minimal amplification by one round trip is:

$$\exp(4\alpha_m L_m)$$

and the round trip phase shift is:

$$\exp(-2j\omega L/c)$$

Then, we get

$$r_1 r_2 \exp(2\alpha_m L_m - j2\omega L/c) = 1.$$

This implies

$$\alpha_m = \frac{1}{2L_M} \ln\left(\left|\frac{1}{r_1}\right| \cdot \left|\frac{1}{r_2}\right|\right)$$

The energy density of the electrical field is (see [2]):

$$\frac{\epsilon}{2}|E|^2$$

Thus, by (1), we obtain

$$n(z) = \frac{\epsilon}{2\hbar\omega}|E|^2 = \frac{\epsilon}{2\hbar\omega}|E_0|^2 \exp(+2\alpha_m z)$$

Since $z = ct$, we obtain

$$n(t) = \frac{\epsilon}{2\hbar\omega}|E|^2 = \frac{\epsilon}{2\hbar\omega}|E_0|^2 \exp(+2\alpha_m ct).$$

By (2), we get

$$K(N_2 - N_1) = 2\alpha_m c \tag{5}$$

Consequences:

$$\begin{array}{rcl} 2\omega L/c & \in & 2\pi\mathbb{Z} \quad \Rightarrow \text{only certain frequencies!} \\ |r_1 r_2| \exp(2\alpha_m L_m) = 1 & \Rightarrow & N_2 - N_1 \geq \dfrac{c}{K}\dfrac{1}{2L_M} \ln\left(\left|\dfrac{1}{r_1}\right| \cdot \left|\dfrac{1}{r_2}\right|\right) \end{array}$$

This is the threshold inversion population (density).

2 Numerical Discretization of a Scalar Rate Equation

Consider the equations (3) and (4) according section 1.4

$$\begin{aligned} \frac{\partial N}{\partial t} &= -\gamma N n \sigma c - \frac{N + N_{tot}(\gamma - 1)}{\tau_f} + R_p(N_{tot} - N) \\ \frac{\partial n}{\partial t} &= N n \sigma c - \frac{n}{\tau_c} + S \end{aligned}$$

The initial values are

$$N(0) = N_0 \quad \text{and} \quad n(0) = n_0.$$

To discretize the unknowns

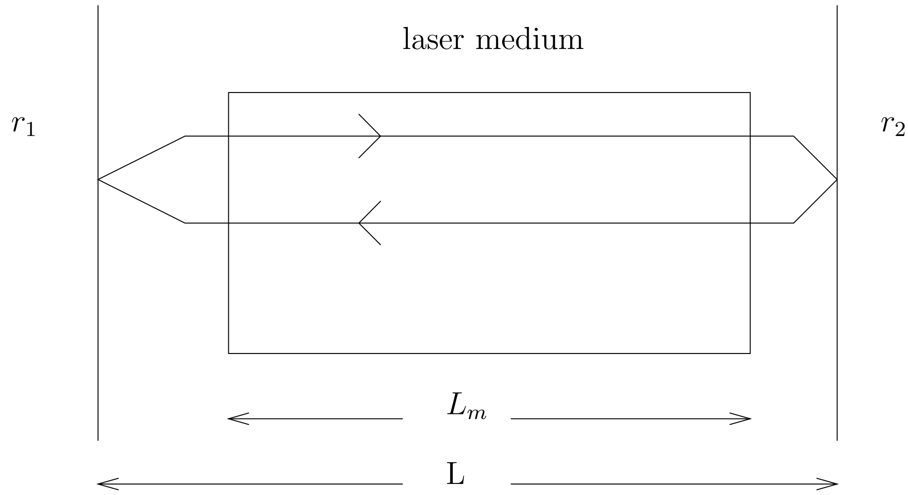


Figure 7: Round trip in laser resonator

- N: population inversion $N = N_2 - N_1$.
- n: photon density

let us use an explicit / implicit Euler discretization with meshsize τ.

Let

- N_s be the approximation of $N(\tau s)$ and
- n_s be the approximation of $n(\tau s)$.

We need a discretization which guarantees that

$$n_s > 0 \quad \text{and} \quad N_s > 0$$

independent of $s \in \mathbb{N}$. Let us assume that $n_s > 0$ and $N_s > 0$ for a fixed s.

- Formula for N_{s+1}: The factor $1/\tau + \gamma * n_s * \sigma * c + 1/\tau_f + R_p$ is positive. Therefore we apply a pure implicit method:

  $N_{s+1} = (N_s/\tau - N_{tot} * (\gamma - 1)/\tau_f + R_p * N_{tot})/(1/\tau + \gamma * n_s * \sigma * c + 1/\tau_f + R_p);$

- Formula for n_{s+1}:

- If $c\sigma N_s - 1/\tau_c > 0$, then we apply an explicit method:

$$n_{s+1} = n_s + \tau * (n_s(c\sigma N_s - 1/\tau_c) + S)$$

- If $c\sigma N_s - 1/\tau_c \leq 0$, then we apply an implicit method:

$$n_{s+1} = (n_s/\tau + S)/(1/\tau - c\sigma N_s + 1/\tau_c)$$

This discretization guarantees that $n_{s+1} > 0$ and $N_{s+1} > 0$. By induction we get $n_s > 0$ and $N_s > 0$ for every $s \in \mathbb{N}$.

Figure 8 depicts a numerical result.

$n(t)$ photon density $N(t)$ population inversion

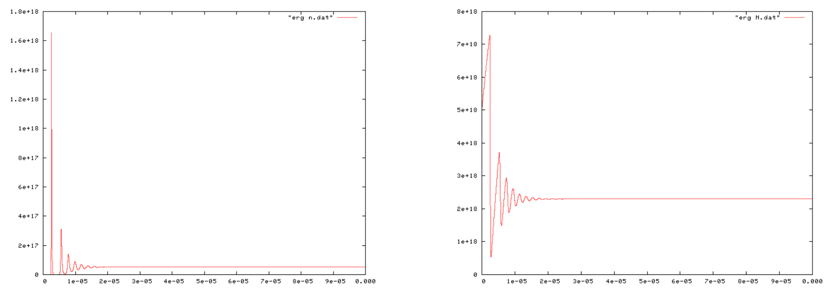


Figure 8: Numerical result

The peak of the photon density after switching on the laser resonator leads to the construction of pulsed lasers.

3 Maxwell's Equations and Helmholtz's Equation

The physical variables of Maxwell's equations are the $3D$-vectors (see [?]):

$$\begin{aligned}
\vec{E} &= \text{electric field intensity (volts / meter)} \\
\vec{D} &= \text{electric field density (coulombs / meter}^2\text{)} \\
\vec{H} &= \text{magnetic field intensity (amperes / meter)} \\
\vec{B} &= \text{magnetic field density (webers / meter}^2\text{)} \\
\vec{J} &= \text{electric current density (amperes / meter}^2\text{)}
\end{aligned}$$

the scalar value

$$\rho = \text{electric charge density (coulombs / meter}^3)$$

and the material parameter

$$\begin{aligned} \epsilon &= \text{permittivity (farads/meter)} \\ \mu &= \text{permeability (henry/meter)} \end{aligned}$$

$$\begin{aligned} \nabla \times \vec{E} &= -\tfrac{\partial \vec{B}}{\partial t} && \text{Faraday's law} \\ \nabla \times \vec{H} &= \tfrac{\partial \vec{D}}{\partial t} + \vec{J} && \text{Maxwell-Ampere law} \\ \nabla \cdot \vec{D} &= \rho && \text{Gauss's law} \\ \nabla \cdot \vec{B} &= 0 && \text{Gauss's law - magnetic} \\ \nabla \cdot \vec{J} &= -\tfrac{\partial \rho}{\partial t} && \text{equation of continuity} \end{aligned}$$

and constitutive relations:

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}, \quad \vec{J} = \sigma \vec{E}$$

By the assumptions:

- μ is roughly constant.
- $\rho = 0$
- $J = 0$

we get

$$\begin{aligned} \nabla \times \vec{E} &= -\tfrac{\partial \vec{B}}{\partial t} && \text{Faraday's law} \\ \nabla \times \vec{H} &= \tfrac{\partial \vec{D}}{\partial t} && \text{Maxwell-Ampere law} \\ \nabla \cdot \vec{D} &= 0 && \text{Gauss's law} \\ \nabla \cdot \vec{B} &= 0 && \text{Gauss's law - magnetic} \\ \vec{D} &= \epsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \end{aligned}$$

Since μ is constant, we get from Maxwell's equations:

$$\begin{aligned} \nabla \times \nabla \times \vec{E} &= -\mu \frac{\partial}{\partial t} \nabla \times \vec{H} \\ &= -\mu \frac{\partial}{\partial t} \left(\frac{\partial \vec{D}}{\partial t} + \vec{J} \right). \end{aligned}$$

Thus, we get

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial^2}{\partial t^2}\left(\epsilon \vec{E}\right) - \mu \frac{\partial}{\partial t}\vec{J}.$$

Now, by $\vec{J} = 0$, we get the vector Helmholtz equation:

$$\boxed{\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial^2}{\partial t^2}\left(\epsilon \vec{E}\right).}$$

Let us assume the ϵ is constant. Then, we get

$$\epsilon \nabla \cdot \vec{E} = \nabla \cdot \vec{D} = \rho = 0.$$

This implies

$$\nabla(\nabla \cdot \vec{E}) = 0. \tag{6}$$

But, ϵ is not constant! Therefore, we assume (6).

Then, we get

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \triangle \vec{E} = -\triangle \vec{E}$$

Furthermore, we assume that
ϵ is constant with respect to time.

Now, the vector-Helmholtz equation

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial^2}{\partial t^2}\left(\epsilon \vec{E}\right).$$

and the assumption (6) imply

$$-\triangle \vec{E} = -\mu \frac{\partial^2}{\partial t^2}\left(\epsilon \vec{E}\right).$$

Assumption (6) is satisfied for the TE-wave (transversal electric wave):

$$\vec{E}(x, y, z) = E(x, y, z)e_x - E(y, x, z)e_y$$

For this wave, we get the scalar Helmholtz equation:

$$\boxed{-\triangle E = -\mu\epsilon \frac{\partial^2}{\partial t^2}(E). \qquad (7)}$$

Let us assume that E is time periodic. This means:

$$E(x, y, z, t) = \exp(i\omega t)E(x, y, z).$$

Inserting in the scalar Helmholtz equation, leads to

$$\boxed{-\triangle E - k^2 E = 0,}$$

where $k^2 = \mu\epsilon\omega^2$.
This is the Helmholtz equation for time periodic solutions.

4 Beam Propagation

4.1 Paraxial Approximation

The paraxial approximation is an approximation of the scalar Helmholtz equation.

$$(\triangle + k^2)E(x, y, z) = 0.$$

Let k_0 be an average value of k. Inserting the ansatz

$$E = e^{-ik_0 z}\Psi(x, y, z)$$

in the scalar Helmholtz equation leads to

$$-\triangle\Psi + 2ik_0\frac{\partial\Psi}{\partial z} + (k_0^2 - k^2)\Psi = 0.$$

In the case that $k = k_0$ is constant, we obtain

$$-\triangle\Psi + 2ik_0\frac{\partial\Psi}{\partial z} = 0.$$

In the paraxial approximation, we neglect the term $\frac{\partial^2\Psi}{\partial z^2}$. This leads to:

$$\boxed{-\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} + 2ik\frac{\partial\Psi}{\partial z} = 0.}$$

4.2 Gauss Mode Analysis

4.2.1 The Lowest Order Gauss-Mode

To solve the paraxial approximation, let us make the ansatz

$$\Psi(x,y,z) = A(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right),$$

where $A(z)$ and $q(z)$ are unknown functions.

This leads to:

$$\begin{aligned}
\frac{\partial\Psi}{\partial x} &= A(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)\left(-ik\frac{2x}{2q(z)}\right)\\
\frac{\partial^2\Psi}{\partial x^2} &= A(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)\left(-k^2\frac{x^2}{q^2(z)}\right)\\
&\quad +A(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)\left(-ik\frac{1}{q(z)}\right)\\
\frac{\partial\Psi}{\partial z} &= A'(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)\\
&\quad +A(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)(-ik(x^2+y^2))(-1)\frac{1}{2q^2}q'.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
0 &= -\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi}{\partial y^2}+2ik\frac{\partial\Psi}{\partial z}\\
&= A(z)\exp\left(-ik\frac{x^2+y^2}{2q(z)}\right)\\
&\quad\cdot\left(k^2\frac{1}{q^2}(x^2+y^2)-k^2\frac{1}{q^2}q'(x^2+y^2)+2ik\frac{1}{q}+2ik\frac{A'}{A}\right)\\
0 &= \frac{k^2}{q^2}(x^2+y^2)(1-q')+2ik\left(\frac{1}{q}+\frac{A'}{A}\right).
\end{aligned}$$

This equation leads to the ODE's

$$\frac{\partial q}{\partial z} = 1 \quad \text{and} \quad \frac{\partial A}{\partial z} = -A\cdot\frac{1}{q}.$$

The unique solutions of these equations are

- $q(z) = q_0 + z$, where q_0 and z_0 are constants.
- $A(z) = A_0 \frac{q_0}{q(z)}$.

Thus, lowest order Gauss mode is

$$\begin{aligned} E(x,y,z) &= e^{-ikz}\Psi(x,y,z) \\ &= A_0 \frac{q_0}{q_0+z} \exp\left(ik\left(-z - \frac{x^2+y^2}{2(q_0+z)}\right)\right) \end{aligned}$$

Let us normalize the amplitude of this mode by $q_0 A_0 = 1$. Then,

$$\boxed{E(x,y,z) = \frac{1}{q_0+z} \exp\left(-ik\left(z + \frac{x^2+y^2}{2(q_0+z)}\right)\right)}$$

Now, let us study the spot size, bream waist and the energy of the lowest order Gauss mode.

Definition 1. *The spot size is defined by the radius r such that*

$$e^{-1} = \frac{|E(z,r)|}{|E(z,0)|}$$

Write

$$\frac{1}{q_0+z} = \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w(z)}$$

where $R(z)$ and $w(z)$ are real valued functions. This means

$$\frac{1}{q_0+z} = (\mathrm{Re}(q_0)+z)\frac{1}{\mathrm{Im}(q_0)^2 + (\mathrm{Re}(q_0)+z)^2} - i\frac{\mathrm{Im}(q_0)}{\mathrm{Im}(q_0)^2 + (\mathrm{Re}(q_0)+z)^2}$$

and

$$\begin{aligned} \frac{1}{R(z)} &= (\mathrm{Re}(q_0)+z)\frac{1}{\mathrm{Im}(q_0)^2 + (\mathrm{Re}(q_0)+z)^2} \\ R(z) &= \frac{\mathrm{Im}(q_0)^2 + (\mathrm{Re}(q_0)+z)^2}{\mathrm{Re}(q_0)+z} \\ &= (\mathrm{Re}(q_0)+z)\left(1 + \frac{\mathrm{Im}(q_0)^2}{(\mathrm{Re}(q_0)+z)^2}\right)^2 \qquad (8) \\ w(z) &= \frac{\lambda}{\pi}\frac{\mathrm{Im}(q_0)^2 + (\mathrm{Re}(q_0)+z)^2}{\mathrm{Im}(q_0)} \end{aligned}$$

$$= \frac{\lambda}{\pi}\left(\mathrm{Im}(q_0) + \frac{(\mathrm{Re}(q_0)+z)^2}{\mathrm{Im}(q_0)}\right) \tag{9}$$

$$\tag{10}$$

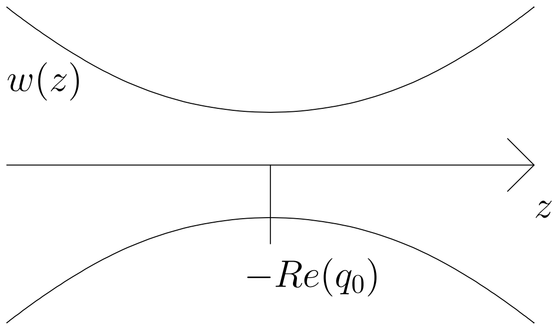


Figure 9: Beam waist of a Gaussian beam.

Phase shift: $\exp\left(-ik\left(z+\frac{x^2+y^2}{2R(z)}\right)\right)$
By this analysis of the Gaussian beam, we get:

- Phase shift: The phase shift of the beam $E(x,y,z)$ behaves like

$$\exp\left(-ik\left(z+\frac{x^2+y^2}{2R(z)}\right)\right)$$

- Spot size: The spot size is $w(z)$.
- Figure 9 shows the beam waist $w(z)$.

Now, let us analyze the energy of the beam at every slice z =constant. To this end, observe that

$$|q_0+z|^2 = \frac{\pi|\mathrm{Im}(q_0)|}{\lambda}|w(z)|$$

$$\begin{aligned}
\int_{\mathbb{R}^2} |\exp(-b(x^2+y^2))|^2 d(xy) &= \int_0^\infty \int_0^{2\pi} \exp(-2br^2) r \; d\varphi dr \\
&= 2\pi \frac{1}{-4b} \exp(-2br^2)\Big|_0^\infty \\
&= \frac{\pi}{2b}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^2} |E|^2 d(xy) &= \left|\frac{A_0 q_0}{q_0+z}\right|^2 \int_{\mathbb{R}^2} \left|\exp\left(-ik\left(\frac{x^2+y^2}{2(q_0+z)}\right)\right)\right|^2 \\
&= \left|\frac{A_0 q_0}{q_0+z}\right|^2 \int_{\mathbb{R}^2} \left|\exp\left(-\frac{\lambda k}{\pi}\frac{(x^2+y^2)}{w(z)}\right)\right|^2 \\
&= \left|\frac{A_0 q_0}{q_0+z}\right|^2 \frac{\pi}{2}\frac{\pi}{\lambda k}|w(z)| \\
&= \frac{|A_0 q_0|^2}{\frac{\lambda |\mathrm{Im}(q_0)|}{\pi}|w(z)|}\frac{\pi}{2}\frac{\pi}{\lambda k}|w(z)| \\
&= \frac{|A_0 q_0|^2}{|\mathrm{Im}(q_0)|}\frac{\pi}{2}\frac{\pi^2}{\lambda^2 k}
\end{aligned}$$

This shows that the energy

$$\int_{\mathbb{R}^2} |E|^2 d(xy) = \frac{|A_0 q_0|^2}{|\mathrm{Im}(q_0)|}\frac{\pi}{2}\frac{\pi^2}{\lambda^2 k}$$

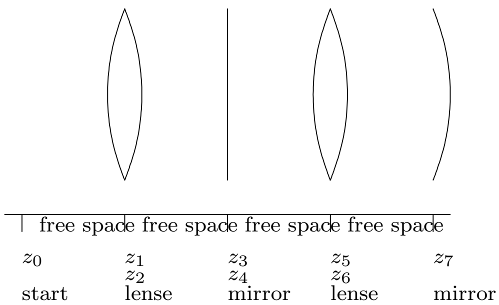
is independent of z.

4.2.2 Gauss Mode in an Aperture

There exists several types of resonators (see Figure 11). Here, let us study a one way resonator. Other resonators can be transformed to a one way resonator.

This means that a beam travels from left to right and that the beam at the right points $z = L$ travels directly to the first point $z = 0$.

Let $\Omega = \Omega_2 \times [0, L]$ be a resonator geometry.
Let us assume that there are n apertures in the resonator.
The start points of these apertures are



$0 = z_0 \leq z_1 \leq z_2 \leq \ldots \leq z_n = L.$

Let us shift the origin of the Gauss-modes in the resonator to these points such that

$$E_i(x,y,z) = \mathcal{A}_i \frac{1}{q_i + (z-z_i)} \exp\left(-ik\left((z-z_i) + \frac{x^2+y^2}{2(q_i+(z-z_i))}\right)\right)$$

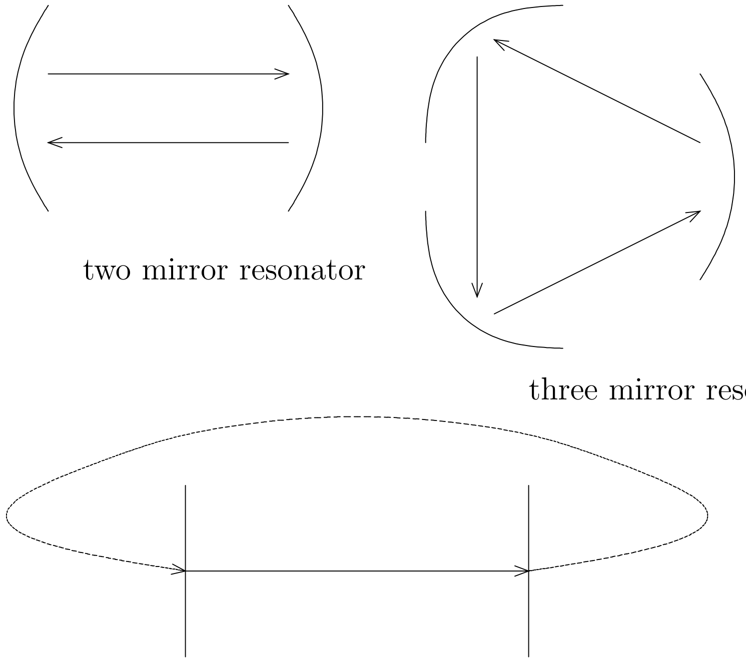


Figure 10: Types of resonators.

where $\mathcal{A}_i := A_i q_i$.

Then, $E_i(x, y, z)$ is the approximation of the electrical field in the subdomain

$$\begin{array}{ccll} \Omega_2 & \times &]z_{i-1}, z_i[& \text{if } z_{i-1} \neq z_i \\ \Omega_2 & \times & [z_{i-1}, z_i] & \text{if } z_{i-1} = z_i \end{array}$$

The change of the Gauss-mode is described by ABCD matrices

$$M_i = \begin{pmatrix} A^i & B^i \\ C^i & D^i \end{pmatrix}$$

Then, the beam parameter q_i changes as follows

$$q_i = \frac{A^i q_{i-1} + B^i}{C^i q_{i-1} + D^i} =: M_i[q_{i-1}].$$

Lemma 1.

$$M_{i+1}[M_i[q_{i-1}]] = (M_{i+1} M_i)[q_{i-1}]$$

This lemma can be proved by a direct calculation.

Another way to prove this lemma is to use that ABCD matrices describe the behavior of rays. To this end, one has to apply the mapping

$$\begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix} \mapsto q = \frac{r_{in}}{r'_{in}}.$$

Then, the above lemma follows by the formula of matrix multiplication.

4.2.3 Ray Optics and ABCD Matrix

Originally, ABCD matrices were used to describe the behavior of rays in optical apertures.

An optical ray can be described by

- the radius $r(z)$ and
- the slope $r'(z)$.

The change of an optical ray is described by

$$\begin{pmatrix} r_{\text{out}} \\ r'_{\text{out}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_{\text{in}} \\ r'_{\text{in}} \end{pmatrix}$$

Example 1 (Ray-matrix of free space)**.**

$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{pmatrix} 1 & \frac{L}{n_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$$

Here, observe that the refraction index is $n_0 = \frac{c}{v}$, where v is the velocity of the optical wave in the medium and c is the velocity in vacuum.

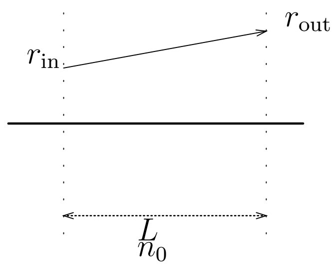


Figure 11: Ray in free space.

4.2.4 ABCD Matrix of free space

By

$$E_i(x,y,z) = E_{i-1}(x,y,z))$$

we obtain

$$\begin{aligned}
&\mathcal{A}_i \frac{1}{q_i + (z - z_i)} \exp\left(-ik\left((z - z_i) + \frac{x^2+y^2}{2(q_i + (z - z_i))}\right)\right) \\
&\quad = \quad \mathcal{A}_{i-1} \frac{1}{q_{i-1} + (z - z_{i-1})} \exp\left(-ik\left((z - z_{i-1}) + \frac{x^2+y^2}{2(q_{i-1} + (z - z_{i-1}))}\right)\right) \\
&\quad \Downarrow \\
&q_i + (z - z_i) = q_{i-1} + (z - z_{i-1}) \\
&\quad \text{and} \quad \mathcal{A}_i \exp(-ik(z - z_i)) = \mathcal{A}_{i-1} \exp(-ik(z - z_{i-1})) \\
&\quad \Downarrow \\
&q_i = q_{i-1} + (z_i - z_{i-1}) \\
&\quad \text{and} \quad \mathcal{A}_i = \mathcal{A}_{i-1} \exp(ik(-(z_i - z_{i-1})))
\end{aligned}$$

This shows

Formula 2 (ABCD matrix of free space)**.**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & z_i - z_{i-1} \\ 0 & 1 \end{pmatrix}$$

and

$$\mathcal{A}_i = \mathcal{A}_{i-1} \exp(ik(-(z_i - z_{i-1})))$$

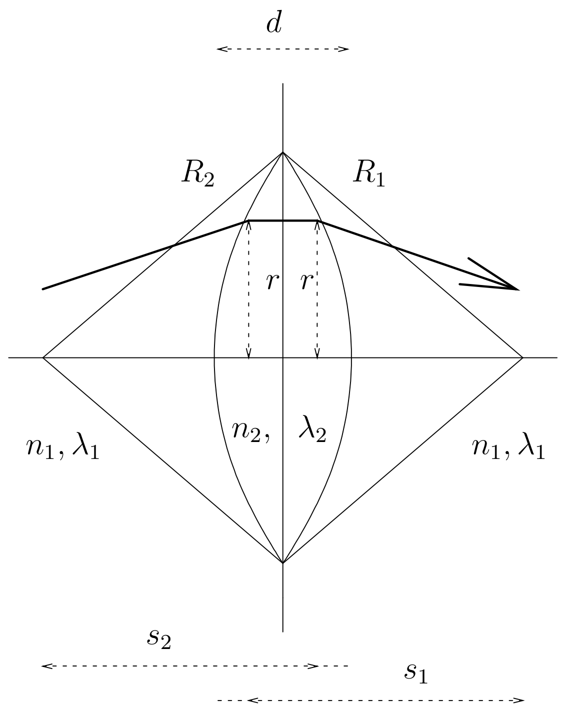


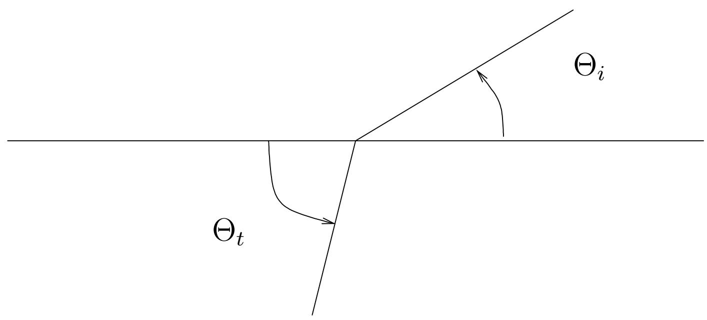
Figure 12: Phase shift of a lense.

4.2.5 ABCD Matrix of a lense

In ray optics the ABCD matrix of a lense of calculated by Snellius law:

$$n_i \sin \Theta_i = n_t \sin \Theta_t,$$

where the angles Θ_i and Θ_t are defined by the following figure:



Let us shift the lense such that $z_{i-1} = z_i = 0$. The lense leads to a phase shift

$$\exp(-i2\pi\varphi(r))$$

where $r = \sqrt{x^2 + y^2}$ such that

$$E_i(x, y, z) = E_{i-1}(x, y, z)) \exp(-i2\pi\varphi(\sqrt{x^2 + y^2})) \tag{11}$$

Let us first calculate this phase shift. By Figure 12, we see that

$$\begin{aligned} s_2^2 + r^2 &= R_2^2 \\ s_1^2 + r^2 &= R_1^2. \end{aligned}$$

Observe the $R_1 + R_2 - d$ is the distance of the to focus points of the lense.

Let us compose the beam by several rays. Then, the length of the way of the ray through the media n_1 is:

$$\begin{aligned} &(R_1 + R_2 - d) - s_1 \;+\; (R_1 + R_2 - d) - s_2 \\ &= \; 2(R_1 + R_2 - d) - s_1 - s_2 \end{aligned}$$

and the length of the way of the ray through the media n_2 is:

$$\begin{aligned} &R_1 + R_2 - d \;-\; (2(R_1 + R_2 - d) - s_1 - s_2) \\ &= \; -(R_1 + R_2 - d) + s_1 + s_2. \end{aligned}$$

To calculate, the phase shift we have to divide by the wavelength λ_1 and λ_2, respectively:

$$\begin{aligned} &\frac{2(R_1 + R_2 - d) - s_1 - s_2}{\lambda_1} + \frac{-(R_1 + R_2 - d) + s_1 + s_2}{\lambda_2} \\ &= \; Q + (s_1 + s_2)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) \\ &= \; Q + (s_1 + s_2)\frac{1}{\lambda_1}\left(\frac{\lambda_1}{\lambda_2} - 1\right), \end{aligned}$$

where Q is a constant term independent of r. Thus, the principal part of the phase shift is contained in

$$(s_1 + s_2)\frac{1}{\lambda_1}\left(\frac{\lambda_1}{\lambda_2} - 1\right)$$

$$
\begin{aligned}
&= \left(\sqrt{R_1^2 - r^2} + \sqrt{R_2^2 - r^2}\right) \frac{1}{\lambda_1} \left(\frac{\lambda_1}{\lambda_2} - 1\right) \\
&\overset{\text{Taylor}}{\approx} \left(R_1 - \frac{1}{2}\frac{r^2}{R_1} + R_2 - \frac{1}{2}\frac{r^2}{R_2}\right) \frac{1}{\lambda_1} \left(\frac{\lambda_1}{\lambda_2} - 1\right) \\
&= \left(R_1 + R_2 - \frac{1}{2} r^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) \frac{1}{\lambda_1} \left(\frac{\lambda_1}{\lambda_2} - 1\right)
\end{aligned}
$$

This shows that the principal part of the phase shift is

$$
\begin{aligned}
\varphi(r) &= -\frac{1}{2}\frac{r^2}{f}\frac{1}{\lambda_1}, \\
\text{where } \tfrac{1}{f} &= \left(\frac{\lambda_1}{\lambda_2} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).
\end{aligned}
$$

Furthermore, we define k to be

$$
k = \frac{2\pi}{\lambda_1}.
$$

Thus, the ansatz (12) leads to

$$
\begin{aligned}
&\mathcal{A}_i \frac{1}{q_i + z} \exp\left(-ik\left(z + \frac{r^2}{2(q_i + z)}\right)\right)|_{z=0} \\
&\quad = \mathcal{A}_{i-1} \frac{1}{q_{i-1} + z} \exp\left(-ik\left(z + \frac{r^2}{2(q_{i-1} + z)}\right)\right) \\
&\quad \cdot \exp\left(-2\pi i\left(-\frac{1}{2}\frac{r^2}{f}\frac{1}{\lambda_1}\right)\right)|_{z=0} \\
&\quad \Downarrow \\
&k\frac{1}{q_i} = k\frac{1}{q_{i-1}} - 2\pi\frac{1}{f}\frac{1}{\lambda_1} \\
&\quad \text{and } \frac{\mathcal{A}_i}{q_i} = \frac{\mathcal{A}_{i-1}}{q_{i-1}} \\
&\quad \Downarrow \\
&\frac{1}{q_i} = \frac{1}{q_{i-1}} - \frac{1}{f} \\
&\quad \text{and } \mathcal{A}_i = \mathcal{A}_{i-1}\frac{q_i}{q_{i-1}} \\
&\quad \Downarrow
\end{aligned}
$$

$$q_i = \frac{q_{i-1}}{-\frac{1}{f}q_{i-1}+1}$$
$$\text{and} \quad \mathcal{A}_i = \mathcal{A}_{i-1}\frac{1}{1-\frac{1}{f}q_{i-1}}$$

This shows

Formula 3 (ABCD matrix of a lense)**.**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \quad \textit{and} \quad \mathcal{A}_i = \mathcal{A}_{i-1}\frac{1}{1-\frac{1}{f}q_{i-1}}$$

Observe that this formula preserves energy, since

$$\frac{|\mathcal{A}_i|}{|\mathrm{Im}(q_i)|} = \frac{|\mathcal{A}_{i-1}|}{|\mathrm{Im}(q_{i-1})|}.$$

(Show this by a calculation as a homework.)

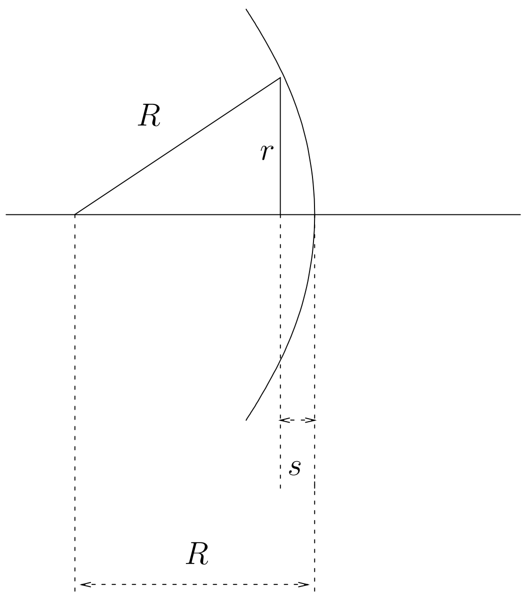


Figure 13: Phase shift of a mirror.

4.2.6 ABCD Matrix of a Mirror

Let us shift the mirror such that $z_{i-1} = z_i = 0$. The mirror leads to a phase shift

$$\exp(-ik(2s(r)))$$

where $r = \sqrt{x^2 + y^2}$ such that

$$E_{i-1}(x, y, z) = E_i(x, y, z) \exp(+ik(2s(r))/\lambda) \tag{12}$$

Here we assume that the wave propagates before and after the mirror in the $+z$ direction.

Let us first calculate this phase shift. By Figure 13, we see that

$$s(r) = R - \sqrt{R^2 - r^2} \overset{\text{Taylor}}{\approx} R - \left(R^2 - \frac{1}{2}\frac{r^2}{R}\right) = \frac{1}{2}\frac{r^2}{R}$$

Thus, we get

$$-ik\frac{r^2}{2(q_{i-1} + z)} = -ik\left(\frac{r^2}{R} + \frac{r^2}{2(q_i + z)}\right)$$

This implies

$$q_i = \frac{q_i + 0}{-q_{i-1}\frac{2}{R} + 1}$$

Formula 4 (ABCD Matrix of a mirror)**.**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$

4.2.7 Other ABCD Matrices

The last two sections showed how to calculate the ABCD matrix of a lense and of free space. Similar calculations lead to the ABCD matrices of other apertures (see [1]). Here, additionally, let us mention the ABCD matrix of a "Gausian Duct":

Formula 5 (ABCD Matrix of a Duct)**.**
Let $k = \omega\sqrt{\mu\epsilon}n(x)$*, where* $n(x) = n_0 - \frac{1}{2}n_2x^2$*. Then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\gamma z) & (n_0\gamma)^{-1}\sin(\gamma z) \\ -(n_0\gamma)\sin(\gamma z) & \cos(\gamma z) \end{pmatrix},$$

where $\gamma^2 = n_2/n_0$*.*

4.2.8 Ray (or Beam) Matrix of the Resonator

The last sections showed how to calculate the ABCD matrix of a lense, mirror and free space.

Using the ABCD matrix M_i of each aperture on can calculate the ABCD matrix of the whole resonator by (see Lemma 1)

$$M = \prod_{i=1}^{n} M_i =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Lemma 2.

$$det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(M) = 1$$

Proof. Observe that for every aperture the corresponding ABCD matrix M_i satisfies $\det(M_i) = 1$. □

Let r_0 be a start vector. Consider

$$r_s = M^s r_0$$

The eigenvalues of M are

$$\lambda_{a,b} := m \pm \sqrt{m^2 - 1}, \quad \text{where } m = \tfrac{A+D}{2}.$$

Observe that $\lambda_a \lambda_b = 1$.

Let q_a, q_b be the eigenvectors of M. Decompose

$$r_0 = c_a q_a + c_b q_b.$$

Such a decomposition is possible, if $q_a \neq q_b$. This is the case of $m \neq 1$.

Then,

$$r_s = c_a \lambda_a^s q_a + c_b \lambda_b^s q_b.$$

- Stable Laser: $-1 \leq |m| \leq 1$. Then,

$$r_s = e^{i\Theta n} c_a q_a + e^{-i\Theta n} c_b q_b,$$

where $\lambda_{a,b} \; := \; \cos\Theta \pm i \sin\Theta = e^{\pm i\Theta}$.

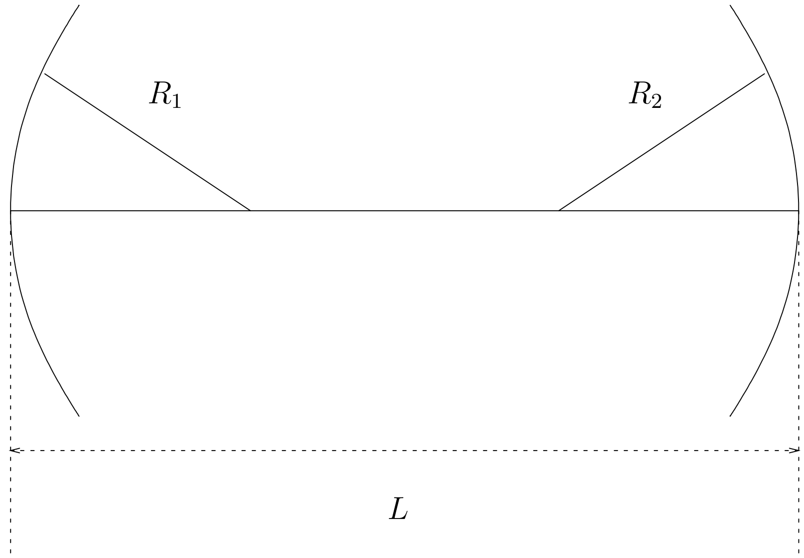


Figure 14: Example of a two mirror resonator.

- Unstable Laser: $|m| \geq 1$. Then,

$$r_s = M^s c_a q_a + M^{-s} c_b q_b,$$

where $M = \lambda_a$, $\frac{1}{M} = \lambda_b$, $M = m + \sqrt{m^2 - 1}$.

Example 2 (Two Mirrors)**.** *Let us assume* $n_0 = 0$. *Consider the resonator in Figure 14 with two mirrors and free space. The corresponding ABCD matrix for* $R_1 = R_2 = R$ *is:*

$$M = \left(\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \right)^2 = \begin{pmatrix} 1 - \frac{2L}{R} & * \\ * & -\frac{2L}{R} + \left(1 - \frac{2L}{R}\right)^2 \end{pmatrix}$$

Let us abbreviate $m = \frac{A+D}{2}$ *and* $\alpha = \frac{2L}{R}$. *Then,*

$$m = 1 - 2\alpha + \frac{1}{2}\alpha^2.$$

Thus the resonator is stable ($|M| \leq 1$*) if and only if* $0 \leq \alpha \leq 4$. *This means*

$$2R \geq L.$$

Figure 15: Gaussian duct.

4.2.9 Exact Solution in a Gaussian "Duct"

The refraction index of a Gaussian duct is (see Figure 15):

$$k = k_0(1 - \frac{1}{2}n_2 r^2)$$

The paraxial approximation and neglecting the small high order term $\frac{1}{4}n_2^2 r^2$ leads to

$$\triangle_{xy}\Psi - 2ik_0\frac{\partial\Psi}{\partial z} - k_0 n_2 r^2 \Psi = 0$$

An exact solution of this equation is:

$$\Psi(x, y, z) = \exp\left(-\frac{x^2 + y^2}{w_1^2} + i\frac{\lambda z}{w_1}\right)$$

where $w_1^2 = 2\frac{1}{k_0\sqrt{n_2}}$ and $\lambda = \frac{2}{k_0}$.

4.2.10 The Guoy Phase Shift

Let us define the Guoy phase shift $\psi(z)$ by:

$$\frac{i|q(z)|}{q(z)} = \exp(i\psi(z)).$$

This implies

$$\tan\psi(z) = \frac{\pi w(z)^2}{R(z)\lambda}.$$

Thus, $\psi(z) = 0$ at the waist of the Gaussian beam.
Then, one can show

$$\frac{1}{w_0}\frac{q_0}{q(z)} = \frac{\exp(i(\psi(z) - \psi_0))}{w(z)},$$

where $\psi_0 = \psi(0)$ and $q_0 = q(0)$.

4.2.11 High Order Modes

Let us the notation in [3]:

In this book the spot size at the waist $z=0$ is:

$$w_D^2(z) \;=\; w_0^2\left(1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2\right)$$

By (9), we get

$$\begin{aligned} w_D^2(z) = w(z)\Big|_{\mathrm{Re}(q_0)=0} &= \frac{\lambda}{\pi}\left(\mathrm{Im}(q_0)+\frac{(\mathrm{Re}(q_0)+z)^2}{\mathrm{Im}(q_0)}\right)\Big|_{\mathrm{Re}(q_0)=0} \\ &\Rightarrow\; w_0^2 = \frac{\lambda}{\pi}\mathrm{Im}(q_0) \end{aligned}$$

and

$$R(z) = (\mathrm{Re}(q_0)+z)\left(1+\frac{\mathrm{Im}(q_0)^2}{(\mathrm{Re}(q_0)+z)^2}\right)^2$$

Hermite-Gaussian Modes

By this notation, we get the Hermite-Gaussian Modes:

$$\begin{aligned} \Psi_{m,n} &= \frac{w_0}{w}H_m\left(\sqrt{2}\frac{x}{w}\right)H_n\left(\sqrt{2}\frac{y}{w}\right) \\ &\quad \exp\left(-i(kz-\Phi)-r^2\left(\frac{1}{w^2}+\frac{ik}{2R}\right)\right) \end{aligned}$$

where

$$\begin{aligned} \Phi(m,n,z) &= (m+n+1)\tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right) \\ H_0(x)=1, &\quad H_1(x)=x, \\ &\quad H_2(x)=4x^2-2, ... \\ &\quad H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \qquad n=0,1,... \end{aligned}$$

The set of these functions forms a basis.

Laguerre-Gaussian Modes

The absolute value of the Laguerre-Gaussian Mode $\Psi_{m,n}$ is:

$$|\Psi_{m,n}| \;=\; E_0\left(\sqrt{2}\frac{r}{w_D}\right)^l L_p^l\left(2\frac{r^2}{w_D^2}\right)\; e^{\frac{r^2}{w_D^2}}\cos(l\phi)$$

where r, ϕ are the angle coordinates and

$$L_0^l(x) = 1 \qquad L_1^l(x) = l + 1 - x$$
$$L_2^l(x) = \frac{1}{2}(l+1)(l+2) - (l+2)x + \frac{1}{2}x^2$$
$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \qquad n = 0, 1, ...$$

The set of these functions forms a basis.

4.2.12 Thermal Lensing

The refraction index $n_c(x)$ of a laser crystal changes by

a) thermal lensing .

b) internal change of the refraction index caused by deformation

c) deformation of the end faces of the laser crystal

a) The refraction index of a laser crystal changes by temperature

- Let T_0 be the temperature before heating (refraction index n_0).
- Let T be the temperature caused by the pumping process (refraction index n).

Let η_T be the thermal index gradient.
(Example: $\eta_T = 2.2 \cdot 10^{-6} \cdot\ ^\circ C^{-1}$ for Cr^{4+}).
Then,

$$n(x, y, z) = n_0 + \eta_T(T(x, y, z) - T_0)$$

The heating of the laser crystal leads to a deformation of the laser crystal. This deformation can be described in the following way.

Let $\mathcal{B} \subset \mathbb{R}^3$ be the original domain of the laser crystal.
Let $T : \mathcal{B} \to \mathbb{R}^3$ be the mapping of the laser deformation such that

$$\{T(x) + x \mid x \in \mathcal{B}\}$$

is the deformed domain of the laser crystal.

- Heat and

- deformation

of the crystal lead to a refraction index

$$n_c(x), \qquad x \in \mathcal{B}$$

such that $k_c(x) = \omega\sqrt{\mu\epsilon}n_c(x)$.

Assume that $\mathcal{B} = \mathcal{D}\times]0, L[$, L length of the laser crystal.

b) The parabolic fit of the refraction index is

- Subdivide $]0, L[$ in N intervals of meshsize $h = \frac{L}{N}$.
- Let $\mathcal{D}_h$ be the discretization grid.
- For every $i = 0, ..., N-1$: Find $n_{0,i}, n_{2,i}$ such that:

$$\left\| n_c(x, y, h(i + \frac{1}{2})) - (n_{0,i} - \frac{1}{2}n_{2,i}(x^2 + y^2)) \right\|_{l^2(\mathcal{D}_h)}$$

- Each of the parameters $n_{0,i}, n_{2,i}$ lead to a matrix

$$A_i = \begin{bmatrix} \cos\gamma_i z & n_0\gamma_i^{-1}\sin\gamma_i z \\ n_0\gamma_i\sin\gamma_i z & \cos\gamma_i z \end{bmatrix}$$

c) Additionally, perform a parabolic fit of $T(x, y, 0)$ and $T(x, y, L)$.

4.3 Beam Propagation Method BPM

The paraxial approximation leads to

$$-\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} + 2ik_0\frac{\partial\Psi}{\partial z} + (k_0^2 - k^2)\Psi = 0.$$

Let us write this equation as follows:

$$2ik_0\frac{\partial\Psi}{\partial z} = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - (k_0^2 - k^2)\Psi.$$

Let $\Omega = \mathcal{D}\times]0, L[$, then one can apply

- FE or FD in x, y-direction

- Crank-Nicolson in z-direction.

Let $\Psi^l(x,y)$ be the approximation of $\Psi(x,y,\tau l)$, where τ is the time step. Then, $\Psi^l(x,y)$ is defined by the equations:

$$\begin{aligned} 2ik_0\frac{\Psi^{l+1}-\Psi^l}{\tau} &= \frac{1}{2}\Bigg(\frac{\partial^2\Psi^{l+1}}{\partial x^2}+\frac{\partial^2\Psi^{l+1}}{\partial y^2}+(k_0^2-k^2)\Psi^{l+1}+ \\ &\qquad \frac{\partial^2\Psi^{l}}{\partial x^2}+\frac{\partial^2\Psi^{l}}{\partial y^2}+(k_0^2-k^2)\Psi^{l}\Bigg) \\ \Psi^0(x,y) &= \Psi^{\text{initial}}(x,y) \qquad\qquad \text{(initial condition)} \end{aligned}$$

- Additional boundary conditions are needed (see section 5.2).

- Lenses and mirrors can be discretized by a phase shift.

4.4 Iteration Method of Fox and Li

Consider a resonator with a left and right mirror. Let Ψ^{initial} be an initial condition at the left mirror. By the BPMethod calculate

- the beam configuration at the right mirror and the
- reflected beam configuration $\Psi^{\text{end}} := \mathcal{B}(\Psi^{\text{initial}})$ at the left mirror.

If $\Psi^{\text{initial}} = \Psi^{\text{end}}$, then Ψ^{initial} is an eigenvector Ψ^{eigen} of the BPM operator $\mathcal{B}$.

The iteration method of Fox and Li is a power iteration method for the eigenvalue problem of the BPM operator $\mathcal{B}$.

This means:

$$\Psi^1 = \Psi^{\text{initial}}, \qquad \Psi^{s+1} = \mathcal{B}(\Psi^{\text{initial,s}})$$
$$\Psi^{\text{eigen}} = \lim_{s\to\infty}\Psi^s$$

The advantages of the BPM and Fox and Li method are:

- 3D approximation
- more general
- simple method

The disadvantages of the BPM and Fox and Li method are:

- no or bad convergence due to several eigenvectors with eigenvalues close to each other.
- bad convergence due to round off errors
- low accuracy
- large computational time
- model errors by the paraxial approximation

5 Finite Element Discretization of Optical Waves in Solid State Laser Resonators

5.1 Weak Formulation of the Helmholtz Equation

Let $\mathcal{D} \subset \mathbb{R}^{d-1}$ be a bounded open domain with a smooth boundary and

$$\begin{aligned} \Omega &= \mathcal{D} \times]0, L[\\ \partial\Omega &= \Gamma_M \dot{\bigcup} \Gamma_R \\ \Gamma_M &= \mathcal{D} \times \{0, L\}. \end{aligned}$$

Let us assume that there are mirrors at 0 and L and non-reflecting boundary conditions at Γ_R. In the Section 5.2, we will show that

$$\begin{aligned} u|_{\Gamma_M} &= 0, \\ u \cdot ik + \frac{\partial u}{\partial n}|_{\Gamma_R} &= 0 \end{aligned}$$

are suitable boundary conditions for the optical wave in a laser resonator. To transform the Helmholtz equation in a weak form, we need suitable spaces.

Let

$$\mathcal{V} := \{v \in H^1(\Omega)|_{\Gamma_M} = 0\}.$$

Then,

$$\begin{aligned} -\triangle u - k^2 u &= f \\ u|_{\Gamma_M} &= 0, \\ u \cdot ik + \frac{\partial u}{\partial n}|_{\Gamma_R} &= 0 \end{aligned}$$

transforms to:

Problem 1. *Find* $u \in \mathcal{V} = \{v \in H^1(\Omega) \mid v|_{\Gamma_M} = 0\}$ *such that*

$$\int_\Omega \nabla u \nabla \bar{v} - k^2 u \bar{v} \, d\mu - ik \int_\Gamma \frac{\partial u}{\partial n} \bar{v} \, d\mu = \int_\Omega f \bar{v} \, d\mu \qquad \text{for every } v \in \mathcal{V}.$$

Define the bilinear form

$$a(u,v) = \int_\Omega \nabla u \nabla \bar{v} - k^2 u \bar{v} \, d\mu - \int_\Gamma \frac{\partial u}{\partial n} \bar{v} \, d\mu$$

Then, the week form of the Helmholtz equation is transforms to:

Problem 2. *Find* $u \in \mathcal{V} = \{v \in H^1(\Omega) \mid v|_{\Gamma_M} = 0\}$ *such that*

$$a(u,v) = \int_\Omega f \bar{v} \, d\mu \qquad \text{for every } v \in \mathcal{V}.$$

Properties of $a(u,v)$:

a) The local part of $a(u,v)$ is the bilinear form

$$a^{\text{loc}}(u,v) = \int_\Omega \nabla u \nabla \bar{v} - k^2 u \bar{v} \, d\mu$$

Let k be constant. Then, a^{loc} is not positive definite, since

$$a^{\text{loc}}(e^{\pm ik_1 z}, e^{\pm ik_1 z}) = \begin{cases} > 0 & \text{if } k_1 > k \\ = 0 & \text{if } k_1 = k \\ < 0 & \text{if } k_1 < k \end{cases}$$

b) Let k be constant. Then, the functions $e^{\pm ikz}$ are contained in the local kernel of a. This means

$$a(e^{\pm ikz}, v) = 0 \quad \text{for every } v \in H^1_0(\Omega).$$

c) The bilinear form $a(u,v)$ is H^1-coercive. This means that there exist $c, C > 0$ such that

$$\text{Re}(a(u,u)) + C\|u\|^2_{L^2} \geq c\|u\|^2_{H^1} \qquad \forall u \in H^1(\Omega)$$

d) The problem

Find $u \in \mathcal{V}$ such that

$$a(u,v) = 0 \qquad \text{for every } v \in \mathcal{V}$$

has the unique solution $u = 0$, if $k > 0$.

The consequence of property d) is, that we cannot model a laser resonator by

$$\begin{aligned} -\triangle u - k^2 u &= 0 \\ u|_{\Gamma_M} &= 0, \\ u \cdot ik + \frac{\partial u}{\partial n}|_{\Gamma_R} &= 0. \end{aligned}$$

Instead, we have to solve the eigenvalue problem

Problem 3. *Find $u \in \mathcal{V} = \{v \in H^1(\Omega)|_{\Gamma_M} = 0\}$ and $\xi \in \mathbb{C}$ such that*

$$a(u,v) = \xi \int_\Omega u\bar{v}\, d\mu \qquad \textit{for every } v \in \mathcal{V}.$$

d) The problem

Find $u \in \mathcal{V}$ such that

$$a(u,v) = 0 \qquad \text{for every } v \in \mathcal{V}$$

has the unique solution $u = 0$, if $k > 0$.

5.2 Boundary Conditions

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be an open d-dimensional open bounded domain. Consider

$$-\triangle u - k^2 u = 0$$

The rays $\exp(ik\, \vec{m} \cdot x)$ are solutions of this equation, where $\vec{m} = 1$.

1D Case:

First, let us consider the 1D case $d = 1$ and $\Omega =]0, 1[$. Then

$$\exp(ikz) \quad \text{and} \quad \exp(-ikz)$$

are solutions of $-\frac{\partial^2 u}{\partial z^2} - k^2 u = 0$.

Let us assume that the reflection of the ray $\exp(-ikz)$ at the point 0 is $\alpha \exp(ikz)$.

This means we need a boundary condition at 0 with solution

$$u(x) = \exp(-ikz) + \alpha \exp(ikz).$$

A suitable boundary condition is

$$u|_{z=0}(1-\alpha)ik + (1+\alpha)\frac{\partial u}{\partial z}|_{z=0} = 0.$$

Thus, a pure *reflecting boundary condition* is

$$\underline{\alpha = -1:} \qquad\qquad u|_{z=0} = 0$$

and a pure *non-reflecting boundary condition* is

$$\underline{\alpha = 0:} \qquad\qquad u|_{z=0} ik + \frac{\partial u}{\partial z}|_{z=0} = 0.$$

2D-3D Case:

The 1D boundary conditions can be generalized to the 2D,3D case as follows:

- *Reflecting boundary condition:*

$$u|_{z=0} = 0$$

- *Non-reflecting boundary condition:*

$$u|_{z=0} ik + \frac{\partial u}{\partial \vec{n}}|_{z=0} = 0.$$

The above non-reflecting boundary condition is not optimal, since the ray $\exp(ik\ \vec{m} \cdot x)$ satisfies the non-reflecting boundary condition, if and only if $\vec{m} = \vec{n}$.

To construct a more accurate boundary condition, it is necessary to extend the computational domain.

<u>Version 1:</u> **Absorption boundary condition by an absorption coefficient in the scalar Helmhotz equation:**

- Observe that

$$\lim_{x \to -\infty} \exp(-i(k+i\alpha)\ \vec{m} \cdot x) = 0,$$

where $\alpha > 0$. This leads to the concept:

- Extend the PDE outside of the domain.
- Add an adsorbtion coefficient α outside of the domain.
- Put homogenous Dirichlet boundary conditions at a certain distance for away from the boundary.

In the general 2D,3D case, define the an additional stripe of size s as follows:

$$S = \{x \notin \Omega \mid \mathrm{dis}(x, \partial\Omega) \leq s\}.$$

This stripe S is used to extend the domain Ω:

$$\Omega \subset \tilde{\Omega} := \Omega \cup S.$$

Observe that $\tilde{\Omega}$ is connected, if Ω is connected.

Now, let us assume that $\mathcal{D}$ is the differential operator on Ω with constant coefficients. Then, define the differential operator

$$\mathcal{D}(u) - (2i\alpha + \alpha^2)u$$

on the stripe S.

Example 3. *Consider the differential operator*

$$-u'' - k^2 u$$

Let us assume non-reflecting boundary conditions at 0 *and reflecting boundary conditions at* 1*. Figure 16 depicts the domain* $\tilde{\Omega} = [-S, 0] \cup [0, 1] = [-S, 1]$ *for these boundary conditions.*

The exact solutions on $[0, 1]$ *are*

$$u(x) = \exp((+/-)ikx)$$

The exact solution on $[-S, 0]$ *are:*

$$u(x) = \exp((+/-)i(k + i\alpha)x)$$

One can show that a small parameter α *and a large stripe width* S *leads to a small reflection of an optical wave at* 0*.*

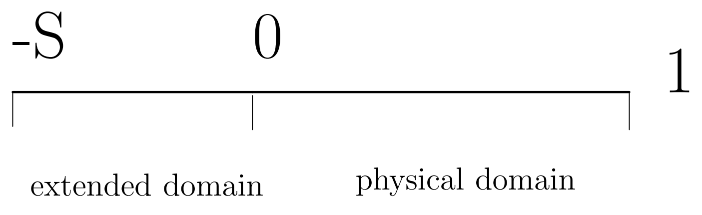


Figure 16: $\tilde{\Omega}$ for $\Omega =]0, 1[$, no reflection at 0 and reflection at 1.

Version 2: Absorption boundary condition by an absorption coefficient in Maxwell equations.
Using the PML (perfect matched boundary layer) method, one can obtain better absorption boundary conditions.

5.3 Difficulties of a Pure Finite Element Discretization

The coercivity of a guarantees that a corresponding eigenvalue problem can be discretized by finite elements. But there are several difficulties:

- One difficulty is the large number of discretization grid points which are needed in case of long resonators. This means that wavelength λ is small in comparison to the resonator length L. Difficulties occur, if $1cm = L >> 5\lambda = 10\mu m$. Then, more than $20 * 1000 = 20000$ grid points are needed only in z-direction.
- The second difficulty is that a is not symmetric positive definite and the resulting linear equation system cannot efficiently be solved by multigrid or any other standard iterative solver.
- There exist several eigenvectors with eigenvalues close to each other.

- A very accurate discretization of the non-reflecting boundary condition is needed.

5.4 Eigenvalue Problem for Long Laser Resonators

5.4.1 Model

Property d) in section 5.1 shows that a simple straight forward model of the wave in a laser resonator does not lead to PDE with a mathematical solution, which is not the trivial solution 0. To improve the straight forward model, we have to derive an appropriate eigenvalue problem. To this end, let us make the ansatz

$$\tilde{E}(x,y,z) = \exp\left[-i(k_f - \varepsilon)z\right] \tilde{u}(x,y,z), \tag{13}$$

where $k_f \in \mathbb{C}$ is an average value of $k(x,y,z)$ and $\varepsilon \in \mathbb{C}$ and $\tilde{u}(x,y,z)$ are unknowns. Furthermore, let us restrict to a one way resonator. This means we assume periodic boundary conditions in the direction of the traveling wave.

Let us abbreviate

$$k_s(x,y,z) = k_f - k(x,y,z). \tag{14}$$

If the variation of $k(x,y,z)$ is small, then $k_s(x,y,z)$ is small in comparison to k_f. Inserting the ansatz (13) into the Helmholtz equation

$$-\triangle E - k^2 E = 0 \tag{15}$$

leads to

$$-\Delta\tilde{u} + 2i(k_f - \varepsilon)\frac{\partial \tilde{u}}{\partial z} + k_s(2k_f - k_s)\tilde{u} = \varepsilon(2k_f - \varepsilon)\tilde{u}. \tag{16}$$

If the variation of $k(x,y,z)$ is small and k_f is an average value of $k(x,y,z)$, then ε is small in comparison to k_f. Therefore, let us model the wave $E(x,y,z)$ in a resonator by the following equations

$$\begin{aligned}
-\Delta u + 2ik_f\frac{\partial u}{\partial z} + k_s(2k_f - k_s)u &= \xi u \\
E(x,y,z) &= \exp\left[-i(k_f - \varepsilon)z\right] u(x,y,z) \\
2\varepsilon k_f &= \xi
\end{aligned}$$

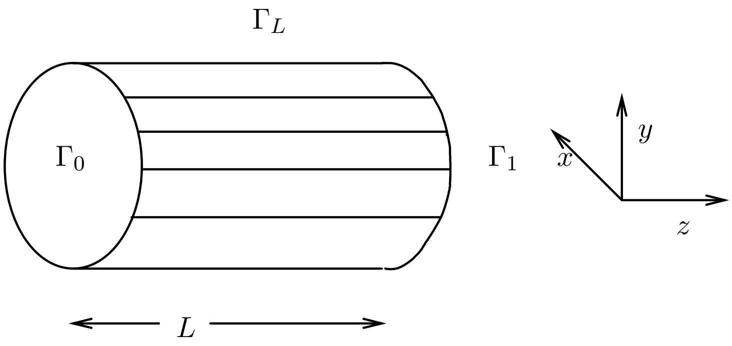


Figure 17: Geometry

5.4.2 Two Wave Eigenvalue Model

Let us assume that $\Phi \subset \mathbb{R}^2$ is a bounded and connected domain with a piecewise smooth boundary and let

$$\Omega = \Phi \times]0, L[,$$

where $L > 0$. Let us subdivide the boundaries of Ω by (see Figure (17)

$$\Gamma_0 := \Phi \times \{0\}, \quad \Gamma_L := \Phi \times \{L\} \text{ and } \Gamma_r := \partial\Omega \setminus (\Gamma_0 \cup \Gamma_L).$$

Reflecting boundary conditions can be modeled by pure Dirichlet boundary conditions at $\Gamma_0 \cup \Gamma_L$

$$E\Big|_{\Gamma_0 \cup \Gamma_L} = 0 \tag{17}$$

and a non-reflecting boundary condition by a Robin boundary condition

$$\frac{\partial \tilde{E}}{\partial \vec{n}} - iC_b\tilde{u}\Big|_{\Gamma_r} = 0, \tag{18}$$

where $\partial/\partial\vec{n}$ denotes the derivation in direction of the normalized outer normal, and $C_b \geq 0$ can be chosen as $C_b = k_f$, see (Ihlenburg[13] and the references cited therein).

For reasons of simplicity, let us additionally assume that we choose k_f such that

$$exp[jLk_f] = 1. \tag{19}$$

To model the eigenmodes of a cavity, it is necessary to apply a two-wave ansatz. These are the forward wave E_r and the backward wave E_l such that

$$E = E_r + E_l,$$

where each of these waves satisfy the Helmholtz equation (15).

Using the ansatz

$$
\begin{aligned}
E_r(x,y,z) &= \exp\left[-j(k_f-\varepsilon)z\right]\tilde{u}_r(x,y,z),\\
E_l(x,y,z) &= \exp\left[-j(k_f-\varepsilon)(L-z)\right]\tilde{u}_l(x,y,z),
\end{aligned}
$$

leads to the eigenvalue problem

$$
\begin{aligned}
-\Delta u_r + 2jk_f\frac{\partial u_r}{\partial z} + (k_f^2-k^2)u_r &= \xi u_r, \qquad (20)\\
-\Delta u_l - 2jk_f\frac{\partial u_l}{\partial z} + (k_f^2-k^2)u_l &= \xi u_l,
\end{aligned}
$$

where

$$
\begin{aligned}
E_r(x,y,z) &= \exp\left[-jk_fz\right]u_r(x,y,z),\\
E_l(x,y,z) &= \exp\left[-jk_f(L-z)\right]u_l(x,y,z),
\end{aligned}
$$

as in Section 5.4.1.

To satisfy the boundary conditions (17) and (18), we need the boundary conditions

$$
\begin{aligned}
u_r+u_l\Big|_{\Gamma_0\cup\Gamma_L} &= 0, \qquad (21)\\
\frac{\partial u_r}{\partial\vec{n}} - jC_bu_r\Big|_{\Gamma_r} &= 0, \qquad (22)\\
\frac{\partial u_l}{\partial\vec{n}} - jC_bu_l\Big|_{\Gamma_r} &= 0. \qquad (23)
\end{aligned}
$$

Observe that (19) is needed to obtain $E_r+E_l\Big|_{\Gamma_0\cup\Gamma_L}=0$ from $u_r+u_l\Big|_{\Gamma_0\cup\Gamma_L}=0$.

To obtain a system of equations with enough equations, we additionally need the boundary condition

$$
\frac{\partial u_r}{\partial z} - \frac{\partial u_l}{\partial z}\Big|_{\Gamma_0\cup\Gamma_L} = 0. \qquad (24)
$$

This boundary condition can be derived by a periodicity argument. To explain this, let us define the band

$$
\mathcal{B} = \Phi\times]-L,L[\Big/\mathcal{P}_L,
$$

where $\mathcal{P}_L$ is the relation which glues $(-L, x), (L, x)$ together for every point $x \in \Phi$. This means $\mathcal{B}$ is a quotient space with respect to $\mathcal{P}_L$. Now, let us define the mapping

$$\begin{aligned} \{(u_l, u_r) \in \mathcal{C}(\bar{\Omega}) \times \mathcal{C}(\bar{\Omega}) \mid u_r + u_l\Big|_{\Gamma_0 \cup \Gamma_L} = 0\} &\rightarrow \mathcal{C}(\mathcal{B}) \\ (u_l, u_r) \mapsto u &= \left(x \mapsto \begin{cases} u_r(x) & \text{if } x \geq 0 \\ -u_l(-x) & \text{if } x < 0 \end{cases} \right) \end{aligned}$$

Physically, this mapping transforms a resonator with a forward and a backward wave in a one way resonator. Then, by the model in Section 5.4.1, we obtain

$$-\Delta u + 2ik_f \frac{\partial u}{\partial z} + k_s(2k_f - k_s)u = \xi u \qquad \text{on } \mathcal{B}. \tag{25}$$

The regularity theory of PDE's shows that $u \in H^2(\Omega)$. This implies the boundary condition (24).

5.5 Weak Formulation

Let us describe the weak formulation of the eigenvalue problem (20) with boundary conditions (21) and (24). To this end, let us define the space:

$$\vec{\mathcal{H}}^1 = \left\{ (u_r, u_l) \in H^1(\Omega) \times H^1(\Omega) \;\middle|\; u_r + u_l|_{\Gamma_0} = 0,\ u_r + u_l|_{\Gamma_L} = 0 \right\}.$$

and the sesquilinear form

$$\begin{aligned} &\vec{a}((u_r, u_l), (v_r, v_l)) = \\ &= \int_\Omega \left(\nabla u_r \nabla \bar{v}_r + (k_f^2 - k^2) u_r \bar{v}_r + 2jk_f \frac{\partial u_r}{\partial z} \bar{v}_r \right) - jC_b \int_{\Gamma_r} u_r \bar{v}_r \\ &\quad + \int_\Omega \left(\nabla u_l \nabla \bar{v}_l + (k_f^2 - k^2) u_l \bar{v}_l - 2jk_f \frac{\partial u_l}{\partial z} \bar{v}_l \right) - jC_b \int_{\Gamma_r} u_l \bar{v}_l, \end{aligned}$$

where we assume that $k \in L^\infty(\Omega)$.

Now, the weak form of the eigenvalue problem (20) with boundary conditions (21) and (24) is

Find $\vec{u} = (u_r, u_l) \in \vec{\mathcal{H}}^1$ and $\xi \in \mathbb{C}$ such that

$$\vec{a}(\vec{u}, \vec{v}) = \xi \int_\Omega u_r \overline{v_r} + u_l \overline{v_l} \qquad \forall \vec{v} = (v_r, v_l) \in \vec{\mathcal{H}}^1.$$

Lemma 3. *Let $C_b = 0$. Then, $\vec{a}(\vec{u}, \vec{v})$ is symmetric.*

But $\vec{a}(\vec{u}, \vec{v})$ is not positive definite.

5.6 Discretization by Finite Elements

For reasons of simplicity, let us assume that $\Psi =]-R, R[\times]-R, R[$. Then, we get

$$\Omega =]-R, R[\times]-R, R[\times]0, L[.$$

Let us discretize this domain by a grid of meshsize $h_x = h_y$ in x- and y-direction and by a grid of meshsize h_z in z-direction. To this end, assume $R/h_x =: N_x \in \mathbb{N}$ and $L/h_z =: N_z \in \mathbb{N}$.

$$\Omega_h := \{(ih_x, jh_y, kh_z) \mid i, j = -N_x, ..., N_x \text{ and } k = 0, ..., N_z\},$$

where we set $h = (h_x, h_y, h_z)$. Furthermore, we obtain the following set of cells

$$\begin{aligned} \tau := \{ \quad & [ih_x, (i+1)h_x] \times [ih_y, (i+1)h_y] \times [ih_z, (i+1)h_z]| \\ & i, j = -N_x, ..., N_x - 1 \quad \text{and} \quad k = -N_z, ..., N_z - 1\}. \end{aligned}$$

Let us define the space of trilinear finite elements by

$$\begin{aligned} V_h := \Big\{ u \in \mathcal{C}(\Omega) \ \Big| \ & \forall T \in \tau : \quad \exists c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \in \mathbb{C} : \\ & u(x, y, z)|_T = c_1 + c_2 x + c_3 y + c_4 z + \\ & c_5 xy + c_6 yz + c_7 xy + c_8 xyz \Big\} \end{aligned}$$

Lemma 4. *For every $\vec{U} = (U_p)_{p \in \Omega_h}$ exists one and only one function $u \in V_h$ such that*

$$u(p) = U_p \quad \forall p \in \Omega_h.$$

Let us define the finite element space

$$\vec{V}_h := \Big\{ (u_{h,r}, u_{h,l}) \in V_h \times V_h \ \Big| \ u_{h,r} + u_{h,l}|_{\Gamma_0} = 0, \ u_{h,r} + u_{h,l}|_{\Gamma_L} = 0 \Big\} \subset \vec{\mathcal{H}}^1$$

Let us first consider the problem:

Let $\vec{f} = (f_r, f_l) \in L^2(\Omega)^2$.
Then, find $\vec{u} = (u_r, u_l) \in \vec{\mathcal{H}}^1$ such that

$$\vec{a}(\vec{u}, \vec{v}) \ = \ \int_\Omega f_r \overline{v_r} + f_l \overline{v_l} \qquad \forall \vec{v} = (v_r, v_l) \in \vec{\mathcal{H}}^1.$$

An unstable FE-discretization is:
Find $\vec{u}_h \in \vec{V}_h$ such that

$$\vec{a}(\vec{u}_h, \vec{v}_h) = \int_\Omega \vec{f}\vec{v}d \quad \forall \vec{v}_h \in \vec{V}_h$$

To explain this instability, let us restrict to the 1D-case and let us consider the limit $k_f \to \infty$. This leads to the bilinear form

$$a_l(u, v) = \int_0^L 2j \frac{\partial u}{\partial z} \bar{v} \ dz$$

The discretization stencil of this bilinear form has the form

$$c \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}.$$

This implies that there is no coupling between the odd and even grid points. In case of a convection diffusion problem this leads to oscillations. In our applications, no observations were observed.

The problem of the above stencil is that the resulting equation system is not diagonal dominant and that there is even no value in the diagonal. Thus, it is difficult to solve the resulting equation system by an iterative solver.

Another problem is that the bilinear form $\vec{a}$ is not positive definite. Thus, the standard finite element theory cannot be applied. Let us recall this theory:

Theorem 2. *Let a be a continuous symmetric positive definite sesquilinear form on a Hilbert space V, V_h a closed subspace and $f \in V'$. Furthermore, let $u \in V$ and $u_h \in V_h$ such that*

$$\begin{aligned} a(u, v) &= f(v) \quad \forall v \in V \\ a(u_h, v_h) &= f(v_h) \quad \forall v_h \in V_h \end{aligned}$$

Then,

$$\|u - u_h\|_E \le \inf_{v_h \in V_h} \|u - v_h\|_E,$$

where $\|.\|_E$ is the norm corresponding to a.

Theorem 3. *Let a be a continuous positive definite sesquilinear form on a Hilbert space V, V_h a closed subspace of V and $f \in V'$. Furthermore, let us assume that a is V-elliptic. This means that there is a constant $\alpha > 0$ such that*

$$|a(u,u)| \geq \alpha \|u\|^2 \quad \forall u \in V.$$

The continuity of a implies that there is a constant C such that

$$a(u,v) \leq C\|u\|\|v\| \quad \forall u,v \in V.$$

Furthermore, let $u \in V$ and $u_h \in V_h$ such that

$$a(u,v) = f(v) \quad \forall v \in V, \qquad a(u_h,v_h) = f(v_h) \quad \forall v_h \in V_h.$$

Then,

$$\|u - u_h\| \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|.$$

Proof. For all $v \in V_h$:

$$\begin{aligned} \alpha\|u-u_h\| &\leq |a(u-u_h, u-u_h)| \\ &= |a(u-u_h, u-v) + a(u-u_h, v-u_h)| \\ &= |a(u-u_h, u-v)| \\ &\leq C\|u-u_h\| \; \|u-v\|. \end{aligned}$$

Therefore,

$$\|u - u_h\| \leq \inf_{v \in V_h} \|u - v\|$$

□

To obtain a stable discretization, we apply the streamline diffusion concept. To this end, let us consider the resonator equation extended to a band (see equation 25) with right hand side f. This is

$$-\Delta u + 2ik_f \frac{\partial u}{\partial z} + k_s(2k_f - k_s)u = f \qquad \text{on } \mathcal{B}. \tag{26}$$

Let us extend the subdivision τ of Ω to a subdivision $\tau_{\mathcal{B}}$ of $\mathcal{B}$ by using the same meshsize. Furthermore, let $V_{h,\mathcal{B}}$ be the corresponding finite element space of trilinear functions.

Now, we can multiply equation (26) by $v + h\rho\frac{\partial}{\partial z}v_h$, where $v_h \in V_{h,\mathcal{B}}$ is a test function and ρ is a constant. Then, we obtain

$$\begin{aligned}\int_T -\Delta u(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h) + 2ik_f\frac{\partial u}{\partial z}(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h) + \\ k_s(2k_f - k_s)u(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h)\ d &= \int_T f(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h)\ d\end{aligned}$$

for every $T \in \tau_{\mathcal{B}}$. A summation for every $T \in \tau_{\mathcal{B}}$ leads to:

$$\begin{aligned}\sum_{T\in\tau_{\mathcal{B}}}\int_T -h\rho\triangle u\nabla\bar{v}_h d - C_b\int_{\partial\mathcal{B}} u\bar{v}_h &+ \\ +\int_{\mathcal{B}} \nabla u\nabla\bar{v}_h + 2ik_f\frac{\partial u}{\partial z}(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h) &+ \\ k_s(2k_f - k_s)u(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h)\ d &= \int_T f(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h)\ d\end{aligned}$$

Observe that $\triangle u_h = 0$ on T for $u_h \in V_{h,\mathcal{B}}$. Now, the streamline-diffusion discretization replaces u by $u_h \in V_{h,\mathcal{B}}$: Discretization: Find $u_h \in V_{h,\mathcal{B}}$ such that

$$\begin{aligned}-C_b\int_{\partial\mathcal{B}} u_h\bar{v}_h &+ \\ +\int_{\mathcal{B}} \nabla u_h\nabla\bar{v}_h + 2ik_f\frac{\partial u_h}{\partial z}(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h) &+ \\ k_s(2k_f - k_s)u_h(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h)\ d &= \int_T f(\bar{v}_h + h\rho\frac{\partial}{\partial z}\bar{v}_h)\ d\end{aligned}$$

$\forall v_h \in V_{h,\mathcal{B}}$.

The term

$$h\rho\int_{\mathcal{B}} k_s(2k_f - k_s)u_h\frac{\partial}{\partial z}\bar{v}_h\ d$$

is very small in this bilinear form. Thus, we can omit this term.

The above discretization on $\mathcal{B}$ leads to the following discretization on Ω: An stable FE-discretization is:

Find $\vec{u}_h \in \vec{V}_h$ such that

$$\begin{aligned}
\vec{a}(\vec{u}_h, \vec{v}_h) &+ \mathbf{h}\rho \int_{\mathbf{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,r}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,r}}}{\partial \mathbf{z}}\ d \\
&+ \mathbf{h}\rho \int_{\mathbf{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,l}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,l}}}{\partial \mathbf{z}}\ d \\
&= \int_\Omega \vec{f} \vec{v}_h d \\
&+ h\rho \left(\int_\Omega f_r \frac{\partial}{\partial z} \bar{v}_{h,r}\ d - \int_\Omega f_l \frac{\partial}{\partial z} \bar{v}_{h,l}\ d \right) \quad \forall \vec{v}_h \in \vec{V}_h
\end{aligned}$$

We call this discretization streamline-diffusion discretization. However, there are no streamlines. In case of a convection-diffusion equation, this discretization converges with $O(h^2)$.

The sesquilinear form of the streamline-diffusion discretization is

$$\begin{aligned}
\vec{a}_h(\vec{u}_h, \vec{v}_h) &= \vec{a}(\vec{u}_h, \vec{v}_h) \\
&+ \mathbf{h}\rho \int_{\mathbf{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,r}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,r}}}{\partial \mathbf{z}}\ \mathbf{d} + \mathbf{h}\rho \int_{\mathbf{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,l}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,l}}}{\partial \mathbf{z}}\ \mathbf{d}
\end{aligned}$$

Lemma 5. *For every $\vec{v}_h \in \vec{V}_h$ the following inequality holds:*

$$|\vec{a}_h(\vec{v}_h, \vec{v}_h)| \geq h k_f \rho \left\| \frac{\partial \vec{v}_h}{\partial z} \right\|^2 .$$

$$\begin{aligned}
\vec{a}_h(\vec{u}_h, \vec{v}_h) &= \vec{a}(\vec{u}_h, \vec{v}_h) \\
&+ \mathbf{h}\rho \int_{\mathbf{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,r}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,r}}}{\partial \mathbf{z}}\ \mathbf{d} + \mathbf{h}\rho \int_{\mathbf{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,l}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,l}}}{\partial \mathbf{z}}\ \mathbf{d}
\end{aligned}$$

Lemma 6. *Let $\vec{u}_h^c \in \vec{\mathcal{H}}^1$ such that:*

$$\vec{a}_h(\vec{u}_h^c, \vec{v}) = \int_\Omega \vec{f} \vec{v}\ d \quad \forall \vec{v} \in \vec{\mathcal{H}}^1.$$

Then,

$$\left\| \frac{\partial^2 \vec{u}_h^c}{\partial z^2} \right\|_2 \leq \frac{C}{h\rho k_f} \|\vec{f}\|_{L^2}.$$

Since $\vec{a}_h$ satisfies the Garding inequality, one can prove the following convergence theorem:

Theorem 4. *Assume $\vec{f} = (f_r, f_l) \in L^2(\Omega)^2$. Let $\vec{u} = (u_r, u_l) \in \vec{\mathcal{H}}^1$ such that*

$$\vec{a}(\vec{u}, \vec{v}) = \int_\Omega f_r \overline{v_r} + f_l \overline{v_l} \qquad \forall \vec{v} = (v_r, v_l) \in \vec{\mathcal{H}}^1.$$

and $\vec{u}_h = (u_{r,h}, u_{l,h}) \in \vec{V}_h$ such that

$$\vec{a}_h(\vec{u}_h, \vec{v}_h) = \int_\Omega f_r \overline{v_r} + f_l \overline{v_l} \qquad \forall \vec{v}_h = (v_r, v_l) \in \vec{V}_h.$$

Then, $\vec{u}_h$ converges to $\vec{u}$.

Instead of

$$\begin{aligned} \vec{a}_h(\vec{u}_h, \vec{v}_h) &:= \vec{a}(\vec{u}_h, \vec{v}_h) \\ &+ \mathbf{h}\rho \int_{\boldsymbol{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,r}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,r}}}{\partial \mathbf{z}} \, \mathbf{d} + \mathbf{h}\rho \int_{\boldsymbol{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,l}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,l}}}{\partial \mathbf{z}} \, \mathbf{d} \end{aligned}$$

one can define

$$\begin{aligned} \vec{a}_h(\vec{u}_h, \vec{v}_h) &:= \vec{a}(\vec{u}_h, \vec{v}_h) \\ &- \mathbf{h}\rho \int_{\boldsymbol{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,r}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,r}}}{\partial \mathbf{z}} \, \mathbf{d} - \mathbf{h}\rho \int_{\boldsymbol{\Omega}} \mathbf{2ik_f} \frac{\partial \mathbf{u_{h,l}}}{\partial \mathbf{z}} \frac{\partial \bar{\mathbf{v}}_{\mathbf{h,l}}}{\partial \mathbf{z}} \, \mathbf{d} \end{aligned}$$

Then, the meaning of $u_{h,r}$ and $u_{h,l}$ changes and the meaning of t and $-t$ in the ansatz

$$E(x, y, z, t) = \exp(i\omega t) E(x, y, z).$$

In 1D the sesquilinear form

$$2ik_f \int_0^1 \frac{\partial u_h}{\partial z} \bar{v}_h \, d + h\rho \frac{\partial u_h}{\partial z} \frac{\partial \bar{v}_h}{\partial z} \, d$$

leads to the stencil

$$ik_f \frac{1}{2} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} + ik_f \frac{1}{h} \rho h \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} = ik_f \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$$

for $\rho = \frac{1}{2}$. This is the FD upwind discretization. An exact solver for the resulting equation system is a Gauss-Seidel relaxation from left to right.

5.7 Iterative Solver

Hackbusch's rule: Consider a singular perturbed problem with parameter $\epsilon \longrightarrow \epsilon_0$. Then, construct an iterative solver such that this solver is an exact solver for ϵ_0 (usually $\epsilon_0 = 0$).

Transform

$$-\Delta u + 2ik_f \frac{\partial u}{\partial z} + k_s(2k_f - k_s)u = \xi u$$

to

$$-\epsilon\Delta u + 2i\frac{\partial u}{\partial z} + \epsilon k_s(2k_f - k_s)u = \epsilon\xi u$$

where $\epsilon = \frac{1}{k_f}$. Then, in 1D, the streamline diffusion discretization stencil for $\epsilon \longrightarrow 0$ is

$$i\,(-1 \quad 1 \quad 0)$$

An exact solver for the corresponding equation system with suitable boundary conditions is a relaxation from left to right. Thus, we used a relaxation from left to right as a preconditioner for GMRES.

The simplest way to solve the eigenvalue problem is to apply an inverse iteration with shift.

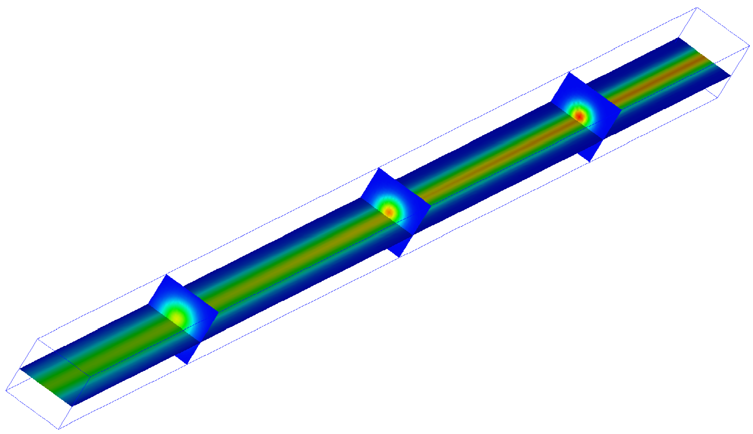


Figure 18: Gauss-Mode by FE

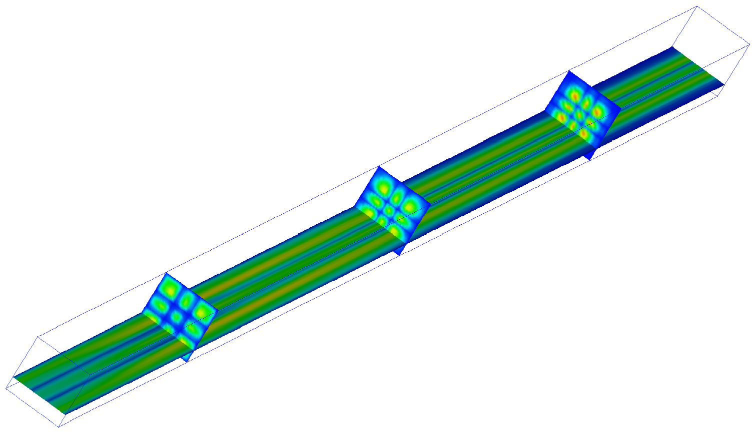


Figure 19: Gauss-Mode by FE

5.8 Modeling of Mirrors, Lenses and Interfaces

Let us assume that there is a lense or an interface at the point l_0 with $0 < l_0 < L$. Furthermore, let us assume a one-way resonator. Then, let us write

$$\Omega_a = \Psi \times [0, l_0] \subset \mathcal{B} \quad \text{and} \quad \Omega_b = \Psi \times [l_0, L] \subset \mathcal{B}. \Psi_I = \Psi \times \{l_0\}.$$

Then, the ansatz

$$E(x, y, z) = \exp\left[-ik_f z\right] u(x, y, z)$$

is not appropriate. Instead, we use the ansatz

$$E(x, y, z) = u(x, y, z) \left\{ \begin{array}{ll} \exp\left[-ik_{f,a} z\right] u(x, y, z) & \text{for } z < l_0 \\ \exp\left[-ik_{f,b} z\right] u(x, y, z) & \text{for } z > l_0. \end{array} \right. ,$$

where $k_{f,a}$ is an average value of k_f in Ω_a and $k_{f,b}$ is an average value of k_f in Ω_b. Let us define the following general bilinear form

$$a_\Xi(u, v) \;\; := \;\; \int_\Xi \nabla u \nabla \bar{v} + 2ik_\Xi \frac{\partial u}{\partial z} \bar{v} + k_s(2k_\Xi - k_s) u \bar{v} \; d$$

and the bilinear form

$$a(u, v) = a_{\Omega_a}(u, v) + a_{\Omega_b}(u, v).$$

From the results in sections 4.2.4 to 4.2.7, we know the phase shift in certain apparatuses. Let us assume that this phase shift is $\varphi(x,y)$. Then, let us define the space

$$\mathcal{H}_{ab} = \left\{ u \in L^2(\mathcal{B}) \,\middle|\, u|_{\Omega_a} \in H^1(\Omega_a), \quad u|_{\Omega_b} \in H^1(\Omega_b) \quad \text{and} \quad u|_{\Psi_I,\Omega_a} \cdot \varphi = u|_{\Psi_I,\Omega_b} \right\}.$$

Here $u|_{\Psi_I,\Omega_c}$ means the restriction of u on Ψ_I as a function in $H^1(\Omega_c)$, $c = a, b$. Now, let us model the wave in the above one way resonator by the following eigenvalue problem Find $u \in \mathcal{H}_{ab}$ such that

$$a(u,v) = \int_{\mathcal{B}} u\bar{v}\; d \quad \forall v \in \mathcal{H}_{ab}.$$

5.9 Gain and Absorption

To simulate gain and absorption in the Helmholtz equation

$$-\triangle u - k^2 u = 0$$

we need the susceptibility $\tilde{\xi}_{at}$ of the medium. According [1] page 267, we get

$$k^2 = \omega^2 \mu\epsilon(1 + \tilde{\xi}_{at} - j\sigma/(\omega\epsilon)).$$

Similar to page 270 in [1], we apply suitable simplifications to obtain

$$k^2 = \omega^2\mu\epsilon + j2\omega\sqrt{\mu\epsilon}\alpha.$$

If $\omega\sqrt{\mu\epsilon}$ is large in comparison to α, then we get

$$k^2 \approx \left(\omega\sqrt{\mu\epsilon} + j\alpha\right)^2$$

Here α is the size of the amplification or gain.

To understand this in more detail, let us consider the 1-dimensional Helmholtz equation

$$-\triangle u - k^2 u = 0.$$

one eigen-solution of this equation is

$$u(z) = \exp(-ikz)$$

Thus, we get

$$|u(z)|^2 = \exp(2\alpha z)$$

and

$$\begin{aligned} E(z,t) &= \exp(i\omega t)\cdot\exp(-i\omega\sqrt{\mu\epsilon}z+\alpha z) \\ &= \exp(i\omega(t-\sqrt{\mu\epsilon}z)+\alpha z). \end{aligned}$$

This shows that u is a wave with

- decreasing amplitude in z direction if $\alpha < 0$ and
- increasing amplitude in z direction if $\alpha > 0$.

Let us write as a gain part and an absorption part

$$\alpha = \alpha_{\text{gain}} - \alpha_{\text{absorption}}$$

$\alpha_{\text{absorption}}$ is mainly a material parameter.

To derive an equation for the gain part, consider the equations (5), (4) and (2)

$$\begin{aligned} KN &= 2\alpha_{\text{gain}}c \\ \frac{\partial n}{\partial t} &= Nn\sigma c - \frac{n}{\tau_c} + S \\ \frac{dn(t)}{dt} &= KN\cdot n(t) \end{aligned}$$

By these equations, we get $K = \sigma c$ and

$$\alpha_{\text{gain}} = \sigma N\frac{1}{2}$$

Thus, we get

$$\begin{aligned} k^2 &= \omega^2\mu\epsilon + j\omega\sqrt{\mu\epsilon}(\sigma N - 2\alpha_{\text{absorption}}) \\ &= \omega^2\mu\epsilon + j\omega\sqrt{\mu\epsilon}(\sigma N - \frac{1}{\tau_c}). \end{aligned}$$

5.10 Time-Dependent Behavior

Using the ansatz

$$E(x,y,z,t) = \exp(i\omega t)(E_r(x,y,z,t) + E_l(x,y,z,t))$$

we obtain

$$
\begin{aligned}
\mu\epsilon\frac{\partial^2 u_r}{\partial t^2} + i\mu\epsilon\omega\frac{\partial u_r}{\partial t} &= \Delta u_r - 2jk_f\frac{\partial u_r}{\partial z} - (k_f^2 - k^2)u_r, \\
\mu\epsilon\frac{\partial^2 u_l}{\partial t^2} + i\mu\epsilon\omega\frac{\partial u_l}{\partial t} &= \Delta u_l + 2jk_f\frac{\partial u_l}{\partial z} - (k_f^2 - k^2)u_l, \\
\frac{\partial N}{\partial t} &= -\gamma N n\sigma c - \frac{N + N_{tot}(\gamma - 1)}{\tau_f} + R_p(N_{tot} - N) \\
k^2 &= \omega^2\mu\epsilon + j\omega\sqrt{\mu\epsilon}(\sigma N - \frac{1}{\tau_c}) \\
n &= \frac{\epsilon}{2\hbar\omega}|E|^2 \\
|E|^2 &= |u_r|^2 + |u_l|^2.
\end{aligned}
$$

5.11 Weak Formulation for the Maxwell Equation

The time-periodic vector Helmholtz equation is

$$\nabla \times \nabla \times \vec{E} - k^2\vec{E} = \vec{f}.$$

The bilinear form of the weak formulation is positive definite:

$$a(\vec{E}, \vec{W}) = \int_\Omega \nabla \times \vec{E} \cdot \nabla \times \bar{\vec{W}} + k^2\vec{E}\bar{\vec{W}}\ d(x, y, z)$$

Let us apply the ansatz

$$
\begin{aligned}
\vec{E} &= e^{-ikz}\vec{u} \\
\vec{W} &= e^{-ikz}\vec{w}.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
a(e^{-ikz}\vec{u}, e^{-ikz}\vec{w}) &= \int_\Omega \nabla \times \vec{u} \cdot \nabla \times \bar{\vec{v}} + (k^2 - k_f^2)u_z\bar{v}_z \\
&\quad -ik_f 2\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\bar{v}_y + (k^2 - k_f^2)u_y\bar{v}_y \\
&\quad -ik_f 2\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)\bar{v}_x + (k^2 - k_f^2)u_x\bar{v}_x\ d(x, y, z)
\end{aligned}
$$

6 Finite Element Discretization of Optical Waves in Semiconductor Laser Resonators

6.1 Construction of Semiconductor Lasers

Semiconductors have different physical properties than solid materials. One of them is that the energy bands in semiconductors are not discrete but a band. To select certain frequencies, most diode lasers use gratings or distributed Bragg reflectors (DBR). A VCSEL (Vertical Cavity Surface Emitting Laser) is depicted in Figure 20 and a DFB laser (Distributed Feedback Laser) in Figure 21.

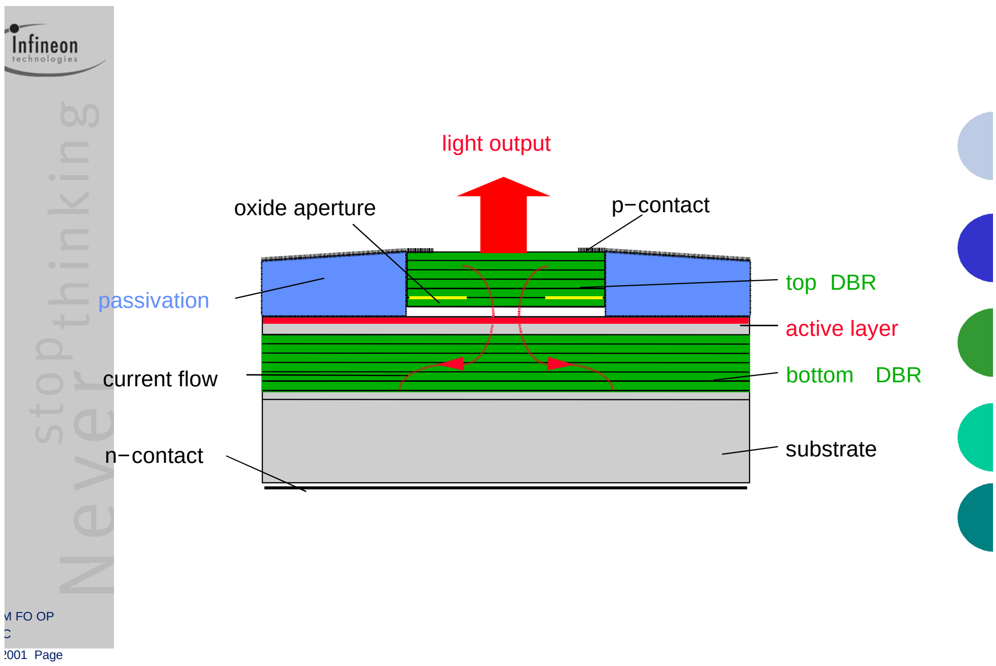


Figure 20: VCSEL (Vertical Cavity Surface Emitting Laser)

6.2 Transfer Matrix Method

To obtain a gain of light for a certain frequency several different constructions are used. The main concept is to use layers of materials with different

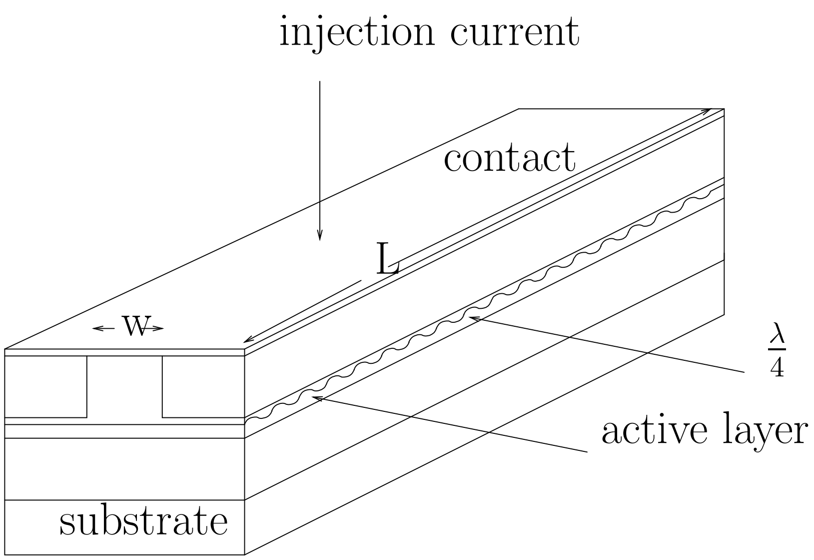


Figure 21: VCSEL (Distributed Feedback Laser)

refraction indices. These layers of different materials form the resonators.

Let us assume that the resonator has the form

$$\Omega = \Psi \times [0, L]$$

and that $0 = l_0 < l_1 < ... < l_s = L$ Furthermore, let us assume that the resonator has the refraction index n_i (k_i) in the layer $\Psi \times [l_{i-1}, l_i]$. Since the layers are very thin, it is important to take into account reflection at the interfaces of the different materials. To understand this in more detail let us consider the 1D case. Assume that

$$-E'' - k^2 E = 0.$$

Let us assume the k is constant in the interior of $[l_{i-1}, l_i]$. Then,

$$E(z) = c_{i,r} \exp(-ik_i(z - l_{i-1})) + c_{i,l} \exp(ik_i(z - l_{i-1})) \quad \text{for } z \in [l_{i-1}, l_i].$$

By the regularity of differential equations, we obtain $E \in C^1([0, L])$. This leads to the following equations at the interfaces (see Figure 22):

$$
\begin{aligned}
c_{i,r}\exp(-ik_i(l_i-l_{i-1}))+c_{i,l}\exp(ik_i(l_i-l_{i-1})) &= c_{i+1,r}+c_{i+1,l}\\
-c_{i,r}\exp(-ik_i(l_i-l_{i-1}))+c_{i,l}\exp(ik_i(l_i-l_{i-1})))k_{i+1})k_i &= (-c_{i+1,r}+c_{i+1,l}).
\end{aligned}
$$

Let us abbreviate $h_i = l_i - l_{i-1}$. Then, we get

$$
\begin{aligned}
&\begin{pmatrix} 1 & 1 \\ -k_i & k_i \end{pmatrix}\begin{pmatrix} \exp(-ik_ih_i) & 0 \\ 0 & \exp(ik_ih_i) \end{pmatrix}\begin{pmatrix} c_{i,r} \\ c_{i,l} \end{pmatrix}\\
&\quad = \begin{pmatrix} 1 & 1 \\ -k_{i+1} & k_{i+1} \end{pmatrix}\begin{pmatrix} c_{i+1,r} \\ c_{i+1,l} \end{pmatrix}\\
&\quad \Downarrow\\
&\begin{pmatrix} c_{i+1,r} \\ c_{i+1,l} \end{pmatrix} = M_i \begin{pmatrix} c_{i,r} \\ c_{i,l} \end{pmatrix}\\
M_i &= \begin{pmatrix} 1 & 1 \\ -k_{i+1} & k_{i+1} \end{pmatrix}^{-1}\\
&\quad \begin{pmatrix} 1 & 1 \\ -k_i & k_i \end{pmatrix}\begin{pmatrix} \exp(-ik_ih_i) & 0 \\ 0 & \exp(ik_ih_i) \end{pmatrix}\\
M_i &= \begin{pmatrix} k_{i+1}+k_i & k_{i+1}-k_i \\ k_{i+1}-k_i & k_{i+1}+k_i \end{pmatrix}\cdot\frac{1}{2k_{i+1}}\begin{pmatrix} \exp(-ik_ih_i) & 0 \\ 0 & \exp(ik_ih_i) \end{pmatrix}.
\end{aligned}
$$

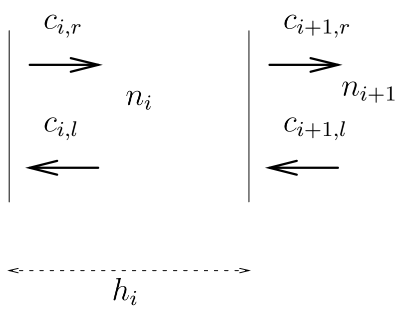


Figure 22: Transmission of two waves from one layer to another layer

In general one can describe the behavior by a scattering matrix S and a transmission matrix T:

$$
\begin{pmatrix} c_{1,r} \\ c_{1,l} \end{pmatrix} = T \begin{pmatrix} c_{2,r} \\ c_{2,l} \end{pmatrix} \qquad \begin{pmatrix} c_{2,r} \\ c_{1,l} \end{pmatrix} = S \begin{pmatrix} c_{1,r} \\ c_{2,l} \end{pmatrix}
$$

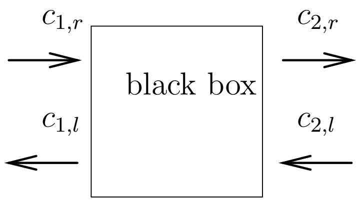


Figure 23: Black box

Example 4. *Let us study* 101 *layers with refraction index* $n_0, n_1, n_0, ..., n_0$, $\lambda_0 = 1.6 \cdot 10^{-6}$, $k_0 = \frac{2\pi}{\lambda_0}$, *and* $\omega = \frac{k}{\sqrt{\epsilon_0 \mu_0} n_0}$, *where* $\sqrt{\epsilon_0 \mu_0} = \frac{1}{c}$ *and* $n_0 = 3.277$. *Let us choose* $c_{2,l} = 1$, $c_{1,r} = 0$. *Then,* $c_{1,l}$ *shows the behavior of the construction. Figure 24 and Figure 25 depict* $c_{1,l}$ *with respect to* ω.

A high reflectivity is obtained for $\omega = \omega_0, 3\omega_0, 5\omega_0, \ldots$.

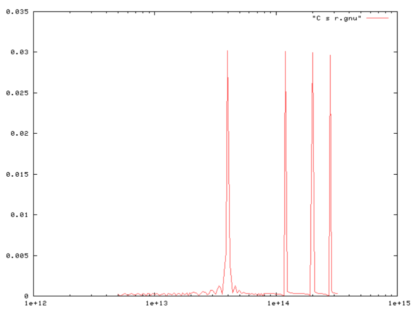


Figure 24: Reflection behavior for $n_1 = 3.275$.

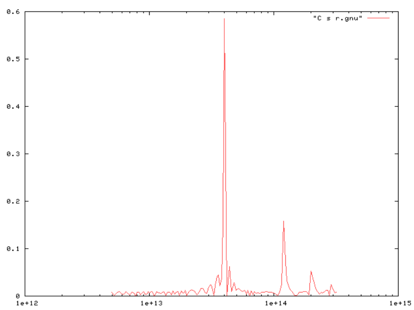


Figure 25: Reflection behavior for $n_1 = 3.220$.

6.3 FE-Discretization for Long Resonators

The two-wave ansatz for solid state laser is not appropriate for semi-conductor lasers, since it does not take into account the reflection property of the medium. Therefore, we construct suitable basis functions, which factor out the high frequency part of the optical wave. Let Ω_h be a grid of meshsize h for the domain $\Omega = [0, L]$. Furthermore, let v_p be the nodal basis function with respect to linear elements. Then, define

$$v_p^l = e^{ikz} v_p$$

$$\begin{aligned} v_p^r &= e^{-ikz} v_p \\ v_p^m &= \begin{cases} e^{ikz} v_p(z) & \text{for } z \leq p \text{ and} \\ e^{-ikz} v_p(z) & \text{for } z > p. \end{cases} \end{aligned}$$

Observe that

$$v_p^l + v_p^r - v_p^m = \begin{cases} e^{-ikz} v_p(z) & \text{for } z \leq p \text{ and} \\ e^{ikz} v_p(z) & \text{for } z > p. \end{cases}$$

Thus, we do not need 4 local basis functions. Now, let us define the FE space

$$V_h^{ref} = \text{span}\{v_p^l, v_p^r, v_p^m \mid p \in \Omega_h\}.$$

This FE space leads to the results as the transfer matrix method. But these basis functions can be extended to 2D and 3D.

Furthermore, the time-dependent scalar Helmholtz equation can be discretized as follows. Let us recall the scalar Helmholtz equation (7):

$$-\triangle \tilde{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2}\left(\tilde{E}\right).$$

The ansatz

$$\tilde{E}(x,y,z,t) = \exp(i\omega t) E(x,y,z,t)$$

leads to

$$\mu\epsilon \frac{\partial^2 E}{\partial t^2} + i\mu\epsilon\omega \frac{\partial E}{\partial t} = \triangle E + \mu\epsilon\omega^2 E.$$

Since ω^2 is large in comparison to $\mu\epsilon$, we apply the following model:

$$i\mu\epsilon\omega \frac{\partial E}{\partial t} = \triangle E + k^2 E.$$

Crank-Nicolson discretization of this equation leads to

$$i\mu\epsilon\omega \frac{E^{s+1} - E^s}{\tau} = \frac{1}{2}\left(\triangle E^s + k^2 E^s + \triangle E^{s+1} + k^2 E^{s+1}\right).$$

Let us analyze this equation by Fourier analysis in 2D. Then, for $E^s = a^s \sin(l_x x)\sin(l_y y)$, we obtain

$$i\mu\epsilon\omega \frac{a^{s+1} - a^s}{\tau} = \frac{1}{2}\left(a^s(l_x^2 + l_y^2 + k^2) + a^{s+1}(l_x^2 + l_y^2 + k^2)\right).$$

This leads to

$$a^{s+1} = \frac{\frac{1}{2}(l_x^2 + l_y^2 + k^2) + i\frac{\mu\epsilon\omega}{\tau}}{\frac{1}{2}(l_x^2 + l_y^2 + k^2) - i\frac{\mu\epsilon\omega}{\tau}} a^s$$

This equation implies

$$|a^{s+1}| = |a^s|$$

if $k \in \mathbb{R}$. This means a real k does not lead to a gain or an absorption. An explicit or implicit Euler discretization does not have this property.

7 Multi-Mode-Analysis

8 Numerical Approximation

Let us assume that $\Omega = \Omega_{2D} \times [0, L]$ is the domain of a laser resonator, where L is the length of the resonator.

Here, let us assume that $E_1, ..., E_M$ are eigenmodes obtained by a Gauss mode analysis or another method. Thus, $E_i : \Omega \to \mathbb{C}$ are functions, which we normalize as follows

$$\int_\Omega |E_i|^2 \, d(x, y, z) = 1.$$

Model Assumption 1 The electrical field E of the total optical wave is approximated by M eigenmodes:

$$E(t, x, y, z) = \sum_{i=1}^{M} \xi_i(t) E_i(x, y, z),$$

where $\xi_i : [t_0, \infty[\to \mathbb{R}$ is the time-dependent coefficient of the i-th mode. Then, the photon density of the mode $\xi_i(t)E_i(x, y, z)$ is

$$n_i(t, x, y, z) \quad = \quad \frac{\epsilon}{2\hbar\omega_i} |\xi_i(t) E_i(x, y, z)|^2 = \frac{\epsilon}{2\hbar\omega_i} \Xi_i(t) |E_i(x, y, z)|^2,$$

where we abbreviate

$$\Xi_i(t) = |\xi_i(t)|^2.$$

ω_i is the frequency of the i-th mode.

Model Assumption 2 The modes are incoherent modes. Here, this means that the total photon density $n(t,x,y,z)$ can be written as

$$n(t,x,y,z)=\sum_{i=1}^{M} n_i(t,x,y,z). \tag{27}$$

Model Assumption 3 The local photon densities $n_i(t,x,y,z)$ and the population inversion density $N(t,x,y,z)$ satisfy the rate equations:

$$\begin{aligned} \frac{\partial n_i}{\partial t} &= N n_i \sigma c - \frac{n_i}{\tau_c} + S, \qquad i=1,..,M, && (28)\\ \frac{\partial N}{\partial t} &= -\gamma N n \sigma c - \frac{N + N_{\mathrm{tot}}(\gamma-1)}{\tau_f} + R_{\mathrm{pump}}(N_{\mathrm{tot}}-N). && (29) \end{aligned}$$

Observe that these rate equations depend on the spatial coordinate (x,y,z) and the time coordinate t. σ is the stimulated emission cross section and c the speed of light. τ_c and τ_f are decay rates. N_{tot} is the concentration of ions per unit volume which are responsible for the laser activity. $R_{\mathrm{pump}}(x,y,z)$ is the pumping rate per unit time per atom at position (x,y,z). σ, τ_c, τ_f, and N_{tot} are constants. But $R_{\mathrm{pump}} : \Omega \to \mathbb{R}$ is a function which describes the pump configuration and the pumping power.

By these model assumptions, we can derive a system of ordinary differential equations, which we can solve numerically. To this end, we insert (27) in (28) and integrate over Ω:

$$\frac{\partial \Xi_i}{\partial t} = \Xi_i \int_\Omega N|E_i|^2 \, d(x,y,z)\, \sigma c - \frac{\Xi_i}{\tau_c} + \frac{2\hbar\omega_i}{\epsilon}\int_\Omega S \, d(x,y,z), \qquad i=1,..,M. \tag{30}$$

Furthermore, we put (27),(27) in (29):

$$\frac{\partial N}{\partial t} = -\gamma N \sigma c \sum_{i=1}^{M} \frac{\epsilon}{2\hbar\omega_i}\Xi_i |E_i|^2 - \frac{N + N_{\mathrm{tot}}(\gamma-1)}{\tau_f} + R_{\mathrm{pump}}(N_{\mathrm{tot}}-N). \tag{31}$$

(30) and (31) form a solvable system of ordinary differential equations, which describes the time-dependent behavior of M modes. This behavior is mainly influenced by the pump configuration R_{pump}.

The solution $(\Xi_i(t))_{i=1,\ldots,M}, N(t,x,y,z)$ can tend to a stationary solution $(\Xi_i^\infty)_{i=1,\ldots,M}, N^\infty(x,y,z)$, which corresponds to the optical wave of a cw-laser.

This stationary solution satisfies the equations

$$
\begin{aligned}
0 &= \Xi_i^\infty \int_\Omega N^\infty |E_i|^2 \; d(x,y,z) \; \sigma c - \frac{\Xi_i^\infty}{\tau_c} + \frac{2\hbar\omega_i}{\epsilon} \int_\Omega S \; d(x,y,z), \qquad i = 1,..,M, \\
0 &= -\gamma N^\infty \sigma c \sum_{i=1}^{M} \frac{\epsilon}{2\hbar\omega_i} \Xi_i^\infty |E_i|^2 - \frac{N^\infty + N_{\text{tot}}(\gamma - 1)}{\tau_f} + \\
&\quad R_{\text{pump}}(N_{\text{tot}} - N^\infty).
\end{aligned}
$$

Motivations for model assumption 2.

Equation (27) is the only crucial point in our model. Therefore, let us present two arguments which motivate assumption 2.

1. Assume that the eigenmodes E_i are orthogonal in the sense

   $$\int_\Omega E_i \bar{E}_j \; d(x,y,z) = \delta_{ij}.$$

   This orthogonality property holds for the Hermite-Gaussian modes[**?**]. Then, (**??**) leads to

   $$\int_\Omega n(t,x,y,z) \; d(x,y,z) = \sum_{i=1}^{M} \int_\Omega n_i(t,x,y,z) \; d(x,y,z),$$

   where

   $$n(t,x,y,z) = \frac{\epsilon}{2\hbar\omega} |E(t,x,y,z)|^2$$

   and ω is an average value of ω_i. This means that (27) in assumption 2 holds in a certain mean value.

2. Assume that the frequencies ω_i are different to each other and that each mode can be represented as

   $$\xi_i(t) E_i(x,y,z) = e^{j\omega_i t} A_i(t) E_i(x,y,z),$$

   where the amplitude $A_i(t)$ consists of small variations. For reasons of simplicity, let us assume that $M = 2$. This means

   $$E(t,x,y,z) = e^{j\omega_1 t} A_1(t) E_1(x,y,z) + e^{j\omega_2 t} A_2(t) E_2(x,y,z).$$

Now, let us choose $\Delta t = 2\pi/(\omega_1 - \omega_2)$.

Since $A_i(t)$ consists of small variations, we obtain

$$\int_t^{t+\Delta t} \xi_1(\tau)E_1(x,y,z)\,\overline{\xi_2(\tau)E_2(x,y,z)}\;d\tau \;\approx$$
$$\approx \int_t^{t+\Delta t} e^{j(\omega_1-\omega_2)\tau}\;d\tau \;\; A_1(t)E_1(x,y,z)\,\overline{A_2(t)E_2(x,y,z)} \;=\; 0.$$

Furthermore, we get

$$\int_t^{t+\Delta t} |\xi_i(\tau)E_i(x,y,z)|^2\;d\tau \;=\int_t^{t+\Delta t} |A_i(\tau)E_i(x,y,z)|^2\,d\tau \approx |A_i(t)E_i(x,y,z)|^2\,\Delta t.$$

The last two equations imply

$$\begin{aligned}\frac{1}{\Delta t}\int_t^{t+\Delta t} |E(\tau,x,y,z)|^2\;d\tau \;&=\; \frac{1}{\Delta t}\int_t^{t+\Delta t} |\xi_1(\tau)E_1(x,y,z)+\xi_2(\tau)E_2(x,y,z)|^2\;d\tau\\ &\approx\; |A_1(t)E_1(x,y,z)|^2 + |A_2(t)E_2(x,y,z)|^2.\end{aligned}$$

This motivates equation (27) of assumption 2.

Obviously, (27) will not hold, if the computed eigenmodes E_i are no physical eigenmodes of the laser. This means E_i are no solutions of Maxwell or Helmholtz equation. Therefore, in the following we assume that the eigenmodes E_i are chosen such that (27) leads to an adequate physical model.

9 Numerical Approximation

Our aim is to solve (30) and (31) numerically. For reasons of simplicity, let us assume that

$$\Omega = [-R,R]^2 \times [0,L]$$

is a cuboid. Now, observe that (30) is a system of ordinary differential equations, which does not depend on a spatial coordinate. But (31) is an ordinary differential equation, which depends on the spatial coordinate (x,y,z). Therefore we discretize (31) by a finite volume discretization. Let Ω_{h_{xy},h_z} be the discretization mesh

$$\Omega_{h_{xy},h_z} = \left\{ \left((i-\frac{1}{2})h_{xy}, (j-\frac{1}{2})h_{xy}, (k-\frac{1}{2})h_z \right) \;\Big|\; i,j = -M_{xy}+1,...,M_{xy}, \;\; k=1,...,M_z \right\},$$

where $h_{xy} = \frac{R}{M_{xy}}$, $h_z = \frac{L}{M_z}$, and $M_{xy}, M_z \in \mathbb{N}$. To every grid point $p = (x, y, z) \in \Omega_{h_{xy},h_z}$ corresponds a discretization cell

$$c_p = \left]x - \frac{h_{xy}}{2}, x + \frac{h_{xy}}{2}\right[\times \left]y - \frac{h_{xy}}{2}, y + \frac{h_{xy}}{2}\right[\times \left]z - \frac{h_z}{2}, z + \frac{h_z}{2}\right[.$$

Observe that

$$\bar{\Omega} = \bigcup_{p \in \Omega_{h_{xy},h_z}} \bar{c}_p.$$

Using a finite volume discretization, we approximate $N(t, x, y, z)$, $(x, y, z) \in c_p$, by the constant value $N_p(t)$ for every point $p \in \Omega_{h_{xy},h_z}$. Then, the finite volume discretization of (30) and (31) leads to

$$\begin{aligned}
\frac{\partial \Xi_i}{\partial t} &= \Xi_i \left(\sum_{p \in \Omega_{h_{xy},h_z}} h_{xy}^2 h_z \; N_p |E_i(p)|^2 \right) \sigma c - \frac{\Xi_i}{\tau_c} + \\
&\quad \frac{2\hbar\omega_i}{\epsilon} \int_\Omega S \; d(x, y, z), \qquad i = 1, .., M, \\
\frac{\partial N_p}{\partial t} &= -\gamma N_p \sigma c \sum_{i=1}^{M} \frac{\epsilon}{2\hbar\omega_i} \Xi_i |E_i(p)|^2 \; \frac{N_p + N_{\text{tot}}(\gamma - 1)}{\tau_f} + \\
&\quad R_{\text{pump}}(p)(N_{\text{tot}} - N_p), \qquad p \in \Omega_{h_{xy},h_z}.
\end{aligned}$$

(32) and (32) form a system of $M + |\Omega_{h_{xy},h_z}|$ scalar ordinary differential equations.

For the time discretization of these equations, we need a stable discretization. A simple but very stable solver is the explicit/implicit Euler discretization. To explain this discretization, let τ be the time step. Observe that the equations (32) and (32) have the form

$$\frac{\partial u}{\partial t} = \lambda(t)u + f,$$

where $f \geq 0$. Furthermore, there is an initial condition $u(t_0) = u_0 \geq 0$. Let us denote $\hat{u}(t_j)$ as an approximation of $u(t_j)$, where $t_j = j\tau + t_0$. Depending on the sign of λ a stable discretization is either the explicit or the implicit Euler discretization:

$$\begin{aligned}
\hat{u}(t_{j+1}) &= \hat{u}(t_j) + \tau(\lambda(t_j)\hat{u}(t_j) + f), && \text{if } \lambda(t_j) > 0, \\
\hat{u}(t_{j+1}) &= (\hat{u}(t_j) + \tau f)(1 - \tau\lambda(t_j))^{-1}, && \text{if } \lambda(t_j) < 0.
\end{aligned}$$

This discretization guarantees that $\hat{u}(t_j) \geq 0$ holds independent of j. Application of this discretization to (32) and (32) leads to approximations $\hat{\Xi}_i(t_j)$ and $\hat{N}_p(t_j)$ for $j = 0, 1, \ldots$.

For reducing the computational time one can apply a stepsize control and suitable high order methods[?].

The finite volume discretization can also be used to calculate an approximation of the stationary equations (32) and (32). To this end, set $\frac{\partial}{\partial t} = 0$ in (32) and (32).

10 Finite Difference Time Domain Method (FDTD) for Maxwell's Equations

10.1 Introduction Maxwell's Equations

The Finite Difference Time Domain Method (FDTD) is an explicit method for the discretization of Maxwell's equations. Therefore, this method is used for the simulation of optical waves.

The solution of Maxwell's equations in 3D is

- $\vec{E}$: the electrical field and
- $\vec{H}$: the magnetic field.

Given are

- μ: magnetic permeability,
- ϵ: electric permittivity,
- $\vec{M}$: equivalent magnetic current density,
- $\vec{J}$: electric current density.

Maxwell's equations are:

$$\begin{aligned} \frac{\partial \vec{H}}{\partial t} &= -\frac{1}{\mu}\nabla \times \vec{E} - \frac{1}{\mu}\vec{M} \\ \frac{\partial \vec{E}}{\partial t} &= \frac{1}{\epsilon}\nabla \times \vec{H} - \frac{1}{\epsilon}\vec{J} \end{aligned}$$

10.2 Finite Difference Time Domain Discretization (FDTD)

Let τ be a time step.

Time approximation:

- $\vec{E}|^{n+\frac{1}{2}}$: approximation at time point $(n+\frac{1}{2})\tau$.
- $\vec{H}|^{n}$: approximation at time point $n\tau$.

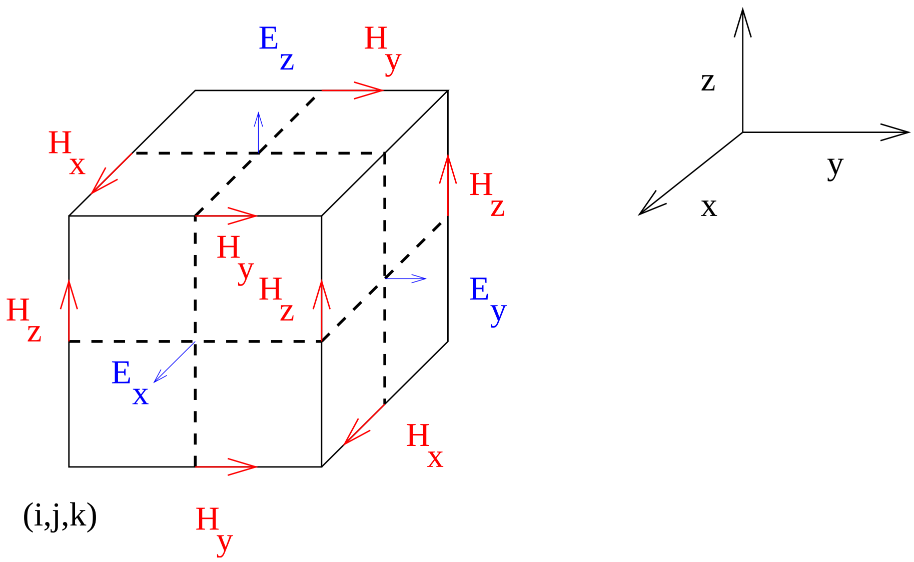
Furthermore, let us use the following abbreviation:

$$
\begin{aligned}
\vec{H}|^{n+\frac{1}{2}} &:= \frac{1}{2}\left(\vec{H}|^{n+1} + \vec{H}|^{n}\right), \\
\vec{E}|^{n} &:= \frac{1}{2}\left(\vec{E}|^{n+\frac{1}{2}} + \vec{E}|^{n-\frac{1}{2}}\right).
\end{aligned}
$$

Let h be a mesh size.

Space approximation:

- $E_x|^{n+\frac{1}{2}}_{i,j+\frac{1}{2},k+\frac{1}{2}}$: at point $(ih,(j+\frac{1}{2})h,(k+\frac{1}{2})h)$ (yz-face) .
- $E_y|^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k+\frac{1}{2}}$: at point $((i+\frac{1}{2})h,jh,(k+\frac{1}{2})h)$ (xz-face).
- $E_z|^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},k}$: at point $((i+\frac{1}{2})h,(j+\frac{1}{2})h,kh)$ (xy-face).
- $H_x|^{n}_{i+\frac{1}{2},j,k}$: at point $((i+\frac{1}{2})h,jh,kh)$ (x-edge).
- $H_y|^{n}_{i,j+\frac{1}{2},k}$: at point $(ih,(j+\frac{1}{2})h,kh)$.
- $H_z|^{n}_{i,j,k+\frac{1}{2}}$: at point $(ih,jh,(k+\frac{1}{2})h)$.



Now, the Maxwell equation

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} + J_x\right)$$

is discretized as follows:

$$\frac{E_x|^{n+\frac{1}{2}}_{i,j+\frac{1}{2},k+\frac{1}{2}} - E_x|^{n-\frac{1}{2}}_{i,j+\frac{1}{2},k+\frac{1}{2}}}{\tau} = \frac{1}{\epsilon_{i,j+\frac{1}{2},k+\frac{1}{2}}} \left(\frac{H_z|^n_{i,j+1,k+\frac{1}{2}} - H_z|^n_{i,j,k+\frac{1}{2}}}{h} - \frac{H_y|^n_{i,j+\frac{1}{2},k+1} - H_y|^n_{i,j+\frac{1}{2},k}}{h} - J_x|^n_{i,j+\frac{1}{2},k+\frac{1}{2}}\right)$$

The other Maxwell's equations are discretized analogously.

This discretization can be written in the following short notation:

Let ∂^1_τ the symmetric difference operator applied to the time coordinate:

$$\partial^1_h Q(t) := \frac{Q(t+\tau/2) - Q(t-\tau/2)}{\tau}$$

Furthermore, let $\nabla_h\times$ the discrete curl operator on a staggered grid. Then the FDTD discretization can be described as follows:

$$\partial^1_\tau \vec{H}_{h,\tau} \;=\; -\frac{1}{\mu}\nabla_h \times \vec{E}_{h,\tau} - \frac{1}{\mu}\vec{M}_{h,\tau} \quad \text{at time points } n+\tfrac{1}{2},$$

$$\partial_\tau^1 \vec{E}_{h,\tau} = \frac{1}{\epsilon}\nabla_h \times \vec{H}_{h,\tau} - \frac{1}{\epsilon}\vec{J}_{h,\tau} \quad \text{at time points } n.$$

Here, $\vec{H}_{h,\tau}$ and $\vec{E}_{h,\tau}$ are the vectors on a staggered grid.

Discretization of Losses and Boundary Conditions

$\vec{J}$ has to be composed as follows:

$$\vec{J} = \vec{J}_{source} + \sigma\vec{E},$$

where σ is the electric conductivity.

$\vec{E}$ is approximated by

$$\vec{E}|^n = \frac{1}{2}\left(\vec{E}|^{n+\frac{1}{2}} + \vec{E}|^{n-\frac{1}{2}}\right).$$

Reflecting boundary conditions can be modeled by pure Dirichlet boundary conditions.

Non-reflecting boundary conditions can be discretized by the Perfect Matched Layer (PML) method. These are not Neumann boundary conditions!

10.3 Stability of FDTD

The Finite Difference Time Domain Method (FDTD) is an explicit method for the discretization of Maxwell's equations. Therefore, this method is used for the simulation of optical waves.

Let us consider the FDTD discretization in the short form for $\vec{J}_{h,\tau} = 0$ and $\vec{M}_{h,\tau} = 0$ and $\mu = 1$ and $\epsilon = 1$:

$$\begin{aligned}\partial_\tau^1 \vec{H}_{h,\tau} &= -\nabla_h \times \vec{E}_{h,\tau} \quad \text{at time points } n+\tfrac{1}{2},\\ \partial_\tau^1 \vec{E}_{h,\tau} &= \nabla_h \times \vec{H}_{h,\tau} \quad \text{at time points } n.\end{aligned}$$

Now, the abbreviation

$$\vec{V}_{h,\tau} = \vec{H}_{h,\tau} + j\vec{E}_{h,\tau}$$

leads to

$$\partial_\tau^1 \vec{V}_{h,\tau} = j\nabla_h \times \vec{V}_{h,\tau}$$

Observe, that $\vec{G}_{h,\tau}$ is a vector defined at all edges and faces of each cell and which is defined at all time points $t\frac{1}{2}n$, where $n \in \mathbb{N}$. To this end set $\vec{H}_{h,\tau}$ and $\vec{E}_{h,\tau}$ to be zero at all points, where these vectors originally are not defined.

Definition 2. *The FDTD method is stable, if the solution $\vec{H}_{h,\tau}, \vec{E}_{h,\tau}$ is bounded for $t \to \infty$.*

To analyze the stability of the FDTD method, we analyze the stability of Let us analyze

$$\partial_\tau^1 \vec{V}_{h,\tau} = j\nabla_h \times \vec{V}_{h,\tau}.$$

To this end, it is enough to analyze the behavior of the solutions with periodic initial condition:

$$\vec{V}_{h,\tau}(0,x,y,z) = \vec{V}_0 e^{j(-k_x x - k_y y - k_z z)}. \tag{32}$$

The FDTD method is stable, if $\vec{V}_{h,\tau}$ has the form

$$\vec{V}_{h,\tau}(t,x,y,z) = \vec{V}_0 e^{j(\omega t - k_x x - k_y y - k_z z)}$$

for every edge point, face point, and every time step $t\frac{1}{2}n$. Observe, that a Fourier decomposition with periodic functions as in the ansatz (32) spans the whole space of possible initial conditions, since every unknown of the vectors $\vec{H}_{h,\tau}$ and $\vec{E}_{h,\tau}$ is located at a different spatial point.

The abbreviation $\vec{V}_0 = (V_x, V_y, V_z)^T$ leads to

$$\begin{aligned}
\nabla_h \times \vec{V}_{h,\tau} &= \det \begin{pmatrix} e_x & \delta^1_{h_x,x} & V_x \\ e_y & \delta^1_{h_y,y} & V_y \\ e_z & \delta^1_{h_z,z} & V_z \end{pmatrix} e^{j(\omega t - k_x x - k_y y - k_z z)} \\
&= \det \begin{pmatrix} e_x & \frac{1}{h_x}\sin(\frac{k_x h_x}{2}) & V_x \\ e_y & \frac{1}{h_y}\sin(\frac{k_y h_y}{2}) & V_y \\ e_z & \frac{1}{h_z}\sin(\frac{k_z h_z}{2}) & V_z \end{pmatrix} e^{j(\omega t - k_x x - k_y y - k_z z)} \\
&= -j\delta^1_\tau \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} e^{j(\omega t - k_x x - k_y y - k_z z)} \\
&= -j\frac{1}{\tau}\sin(\frac{\omega\tau}{2}) \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} e^{j(\omega t - k_x x - k_y y - k_z z)}
\end{aligned}$$

The above equation system has a unique solution if and only if

$$
\begin{aligned}
0 &= \det \begin{pmatrix} j\frac{1}{\tau}\sin(\frac{\omega\tau}{2}) & \frac{1}{h_z}\sin(\frac{k_z h_z}{2}) & -\frac{1}{h_y}\sin(\frac{k_y h_y}{2}) \\ \frac{1}{h_z}\sin(\frac{k_z h_z}{2}) & j\frac{1}{\tau}\sin(\frac{\omega\tau}{2}) & -\frac{1}{h_x}\sin(\frac{k_x h_x}{2}) \\ -\frac{1}{h_y}\sin(\frac{k_y h_y}{2}) & +\frac{1}{h_x}\sin(\frac{k_x h_x}{2}) & j\frac{1}{\tau}\sin(\frac{\omega\tau}{2}) \end{pmatrix} \\
&= \left(\left(\frac{1}{h_x}\sin(\frac{k_x h_x}{2})\right)^2 + \left(\frac{1}{h_y}\sin(\frac{k_y h_y}{2})\right)^2 + \left(\frac{1}{h_z}\sin(\frac{k_z h_z}{2})\right)^2 \right. \\
&\quad \left. - \left(\frac{1}{\tau}\sin(\frac{\omega\tau}{2})\right)^2 \right) j\frac{1}{\tau}\sin(\frac{\omega\tau}{2})
\end{aligned}
$$

. This is equivalent to the *stability equation:*

$$
\left(\frac{1}{h_x}\sin(\frac{k_x h_x}{2})\right)^2 + \left(\frac{1}{h_y}\sin(\frac{k_y h_y}{2})\right)^2 + \left(\frac{1}{h_z}\sin(\frac{k_z h_z}{2})\right)^2 = \left(\frac{1}{\tau}\sin(\frac{\omega\tau}{2})\right)^2
$$

The *stability equation* has a solution ω for every k_x, k_y, k_z, if

$$
\tau\sqrt{\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}} < 1.
$$

A renormalization of this stability condition shows

$$
\tau < c^{-1}\left(\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right)^{-\frac{1}{2}}.
$$

where c is the velocity of the wave.

References

[1] Siegman *Lasers*. University Science Books Sausalito, California.

[2] R. Kröger and R.Unbehauen *Technische Elektrodynamik*. Teubner Stuttgart, 1987.

[3] Christopher C.Davis *Lasers and Electro-Optics (Fundamentals and Engineering)* Cambridge University Press.

[4] Coldren and Corzine *Diode Lasers and Photonic Integrated Circuits* Wiley

[5] Amnon Yariv *Introduction to Optical Electronics.*

[6] Amnon Yariv *Optical Waves in Crystals.*

[7] Koechner *Solid-State Laser Engineering.*

[8] Hecht *Optik.*

[9] Braess *Finite Elemente* Springer.

[10] Ihlenburg *Finite Element Analysis of Acoustic Scattering.* Springer.

[11] Jianming Jin *The Finite Element Method in Electromagnetics.* John Wiley & Sons.

[12] Altmann, Pflaum, Seider. *3D Finite Element Computation of Laser Cavity Eigenmodes.* Applied Optics-LPE . vol. 43, issue 9, page 1892, 2004.

[13] F. Ihlenburg. *Finite Element Analysis of Acoustic Scattering*, volume 132 of *Springer Series Applied Mathematical Sciences.* Springer, New York, Paris, London, 1998.